On the Formalization of Cardinal Points of Optical Systems

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Abstract. Optical systems are widely used in safety critical applications such as aerospace, telecommunication and biomedical systems. The verification of such systems is usually performed by informal techniques (e.g., numerical simulation and paper-and-pencil based proofs) which may result in erroneous designs. Interactive theorem proving has the potential to verify complex optical designs with better accuracy and soundness. However, existing formalizations of optics theories do not provide the facility to analyze optical imaging properties which are used to characterize the behavior of objects under observation (e.g., cancer cells, human eye or commercial camera lenses). In this paper, we present the formalization of cardinal points which are the most fundamental requirement to model imaging properties. We also present the formal verification of the cardinal points for an arbitrary optical system consisting of any number of optical components. In order to demonstrate the usefulness of our formalization, we present the formal analysis of an optical instrument used to compensate the ametropia of an eye.

Keywords: Theorem Proving, HOL Light, Optical Systems, Cardinal Points.

1 Introduction

Generally, optical systems consist of a combination of reflecting and refracting surfaces (i.e., mirrors or lenses) to achieve different functionalities such as astronomical imaging, light modulation and short pulse generation. Modeling and analysis of such systems is based on different abstractions of light such as geometrical, wave, electromagnetic and quantum optics. Geometrical or ray optics [20] characterizes light as a set of straight lines which linearly traverse through an optical system. Wave [26] and electromagnetic optics [26] describe the scalar and vectorial wave nature of light, respectively. In quantum optics [8], light is considered as a stream of photons and electric and magnetic fields are modeled as operators. In general, each of these theories has been used to model different aspects of the same or different optical components. A phase-conjugate mirror [15] can be modeled using the ray, electromagnetic and quantum optics. The application of each theory is dependent on the type of system properties which needs to be verified. For example, ray optics provides a convenient way to verify

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the stability of optical resonators, coupling efficiency of optical fibers and optical imaging of commercial lenses. On the other hand, ensuring that no energy is lost when light travels through a waveguide and the analysis of active elements require electromagnetic and quantum optics theories, respectively. In practice, one of the primary design choices is to model a given optical system using the ray optics theory which provides useful information about the overall structure of the system. Moreover, it provides a convenient way to analyze some important properties describing the transformation of input ray (object ray) to the output ray (image ray). Some of these properties are the optical power of each component, image size and location etc. These properties are called the imaging properties of optical systems which are usually described in terms of cardinal points [26] (i.e., three pair of points on the optical axis which are sufficient to completely specify the imaging properties of most widely used optical systems). Most of the industrial optical system analysis software products (e.g., Zemax [19]) provide the facility to analyze such properties.

One of the most challenging requirement in the validation of the practical optical system models is the verification of desired properties. Therefore, a significant portion of time is spent finding design bugs in order to build accurate optical systems. Traditionally, the analysis of optical systems has been done using paper-and-pencil proofs [26]. However, considering the complexity of optical and laser systems, this analysis is very difficult, risky and error-prone. Many examples of erroneous paper-and-pencil proofs are available in the literature of optics (e.g., work reported in [7] was latter corrected in [18]). Another approach is to perform a simulation-based analysis of optical systems. This is mainly based on numerical algorithms and suffers from numerical precision and soundness problems. The above mentioned inaccuracy problems of traditional analysis techniques are impeding their usage in designing safety-critical optical systems, where minor bugs can lead to disastrous consequences such as the loss of human lives (e.g., surgeries [16]) or financial loss (e.g., the Hubble Telescope [1], for which the total budget was $1.6 billion). In order to build reliable and accurate optical systems, it is indispensable to develop a framework which is both accurate and scalable for handling complex optical and laser systems.

Formal methods [27] allow for accurate and precise analysis and has the potential to overcome the above mentioned limitations of traditional approaches. The main idea behind them is to develop a mathematical model for the given system and analyze this model using computer-based mathematical reasoning, which in turn increases the chances for catching subtle but critical design errors that are often ignored by traditional techniques. In order to formally verify electronic systems, several formal methods based techniques (such as model checking [5] and theorem proving [12]) have been proposed. Due to the involvement of multivariate calculus (complex linear algebra, complex geometry theory) in the design of optical systems, model checking is not suitable to handle such systems. Recently, some preliminary works for analyzing optical systems using theorem proving have been reported in the open literature. For instance, in [14], the formal analysis of optical waveguides using real analysis of HOL4 theorem prover is reported. In [4], complex formalization of electromagnetic optics is reported. The formalization of quantum mechanics and quantum optics is presented in [17] with applications in quantum computing. The preliminary formalization of ray optics is reported in [22,23] with main applications in the analysis of optical and laser resonators [21]. Despite of the vast applications of optical imaging systems, none of the above mentioned work provides the formalization of basic building-blocks such as the notion of cardinal points [26].

The main focus of this paper is to bridge the above mentioned gap and strengthen the formal reasoning support in the area of optical imaging systems. The work presented in this paper is an extension of [25] where we elaborate more on the formalization framework for imaging optical systems along with the formal analysis of an arbitrary visual optical system to verify its effect for refractive compensation when placed in front of a human eye [10]. This work is a part of an ongoing project[1] to develop a formal reasoning support for different fields of optics (e.g., ray, electromagnetic and quantum optics). In this paper, we use the HOL Light theorem prover [3] to formalize the underlying theories of imaging optical systems. The main reasons of using HOL Light is the existence of rich multivariate analysis libraries [13,9]. Our HOL Light developments of geometrical optics and optical imaging systems presented in this paper are available for download [24] and thus can be used by other researchers and optics engineers working in industry to conduct the formal analysis of more practical optical systems.

The rest of the paper is organized as follows: Section 2 and 3 provide a brief introduction of ray optics and the HOL Light theorem prover, respectively. In Section 4, we present the proposed formalization framework for ray optics and optical imaging properties along with some highlights of the formalization of optical systems, rays and corresponding matrix models. We describe the formalization of composed optical systems in Section 5. Consequently, we present the formalization of cardinal points of optical imaging systems and the development of component library in Sections 6 and 7, respectively. We illustrate the effectiveness of our work by describing the formal modeling and analysis of a visual optical system in Section 8. Finally, Section 9 concludes the paper and highlights some future research directions.

2 Ray Optics

2.1 Overview

Ray optics describes the propagation of light as rays through different interfaces and mediums. The main governing principle of ray optics is based on some postulates which can be summed up as follows: Light travels in the form of rays emitted by a source; an optical medium is characterized by its refractive index; light rays follow the Fermat's principle of least time [20]. Generally, the main

[1] `http://hvg.ece.concordia.ca/projects/optics/`

components of optical systems are lenses, mirrors and propagating mediums which is either a free space or some material such as glass. These components are usually centered about an optical axis, around which rays travel at small inclinations (angle with the optical axis). Such rays are called *paraxial rays* and this assumption provides the basis of *paraxial optics* which is the simplest framework of geometrical optics. When a ray passes through optical components, it undergoes *translation*, *refraction* or *reflection*. In translation, the ray simply travels in a straight line from one component to the next and we only need to know the thickness of the translation. On the other hand, refraction takes place at the boundary of two regions with different refractive indices and the ray obeys the law of refraction, called *Paraxial Snell's law* [20]. Similarly, a ray follows the law of reflection at the boundary of a reflective interface (e.g., mirror). For example, ray propagation through a free space of width d with refractive index n, and a plane interface (with refractive indices n_0 and n_1, before and after the interface, respectively) is shown in Figure 1.

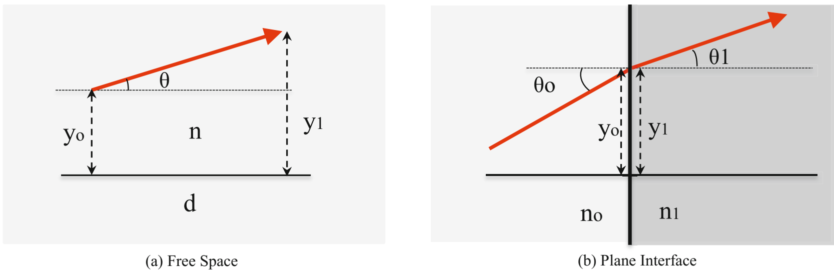


Fig. 1. Behavior of a Ray at Plane Interface and Free Space

2.2 Modeling Approach

The change in the position and inclination of a paraxial ray as it travels through an optical system can be described by the use of a matrix algebra. This matrix formalism (called *ray-transfer matrices*) of geometrical optics provides accurate, scalable and systematic analysis of real-world complex optical and laser systems. This is because of the fact that each optical component can be described by a (2×2) matrix and many linear algebraic properties can be used in the analysis of optical systems. For example, the general optical system with an input and output ray vector can be described as follows:

$$\begin{bmatrix} y_n \\ \theta_n \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} y_0 \\ \theta_0 \end{bmatrix}$$

Finally, if we have an optical system consisting of N optical components (C_i), then we can trace the input ray R_i through all optical components using the composition of matrices of each optical component as follows:

$$R_o = (C_k.C_{k-1}....C_1).R_i \tag{1}$$

We can write $R_o = M_s R_i$, where $M_s = \prod_{i=k}^{1} C_i$. Here, R_o is the output ray and R_i is the input ray. Similarly, a composed optical system that consists of N optical systems inherits the same properties as of a single optical component. This is a very useful modeling notion for the systems which consist of small subsystems due to the already available infrastructure which can be utilized directly with minimal efforts.

2.3 Optical Imaging

Optical systems capable of being utilized for imaging (can record or transform objects to an image) are called optical imaging systems. Mainly these systems are divided into two main categories, i.e., mirror-systems (also called *catoptrics*, which deal with reflected light rays) and lens-systems (also called *dioptrics*, which deal with refracted light rays). Examples of such systems are optical fibers and telescopes, for the first and second case, respectively. An optical imaging system has many cardinal points which are required to analyze imaging properties (e.g., image size, location, and orientation, etc.) of the optical systems. These points are the *principal points*, the *nodal points* and the *focal points*, which are situated on the optical axis. Figure 2 describes a general optical imaging system with an object point P_0 with a distance x_0 from the optical axis (called the object height). The image is formed by the optical system at point P_1 with a distance x_1 from the optical axis (called the image height). The refractive indices of object space and image space are n and n', respectively. The points F and F' are the foci in the object space and the image space, respectively. The points N and N' are the nodal points in the object and image space. Finally, the points U and U' are the unit or principal points in the object and image space [26].

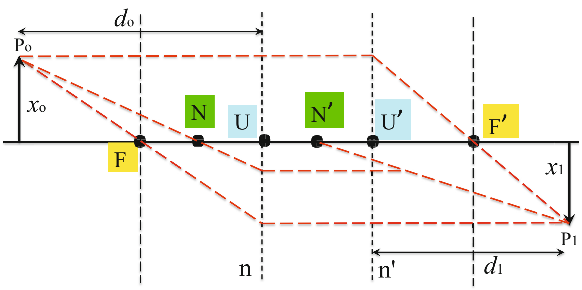


Fig. 2. Cardinal Points of an Optical System [26]

2.4 Ray Tracing

The propagation of paraxial rays through an optical system is a very useful technique to analyse optical systems. The activity of ray propagation through an optical system is called *ray tracing* [26] and it provides a convenient way for the design optimization along with the assessment of imaging quality and properties such as misalignment tolerance and fabrication error analysis of optical components. Ray tracing can be automated and hence it is a part of almost all optical system design tools such as Zemax [19]. There are two types of ray tracing: sequential and non-sequential. In this paper, we only consider sequential ray tracing which is based on the following main modeling criterion [26] :

1. The type of each interface (e.g., plane or spherical, etc.) is known.
2. The parameters of the corresponding interface (e.g., the radius of curvature in the case of a spherical interface) are known in advance.
3. The spacing between the optical components and misalignment with respect to optical axis are provided by the system specification.
4. Refractive indices of all materials and their dependence on wavelength are available.

On the other hand, in case of non-sequential ray tracing the nature of each interface is not predefined, i.e., at each interface, the ray can either be transmitted or reflected. Non-sequential ray tracing is very expensive in terms of its huge computational time and it is only applied when the sequential ray tracing cannot be used. It is sufficient to consider sequential ray tracing to evaluate the performance of most imaging optical systems and hence the main reason of our choice.

3 HOL Light Theorem Prover

HOL Light [11] is an interactive theorem proving environment for the construction of mathematical proofs in higher-order logic. A theorem is a formalized statement that may be an axiom or could be deduced from already verified theorems by an inference rule. A theorem consists of a finite set Ω of Boolean terms called the assumptions and a Boolean term S called the conclusion. For example, "$\forall x. x \neq 0 \Rightarrow \frac{x}{x} = 1$" represents a theorem in HOL Light. A HOL Light theory consists of a set of types, constants, definitions, axioms and theorems. HOL theories are organized in a hierarchical fashion and theories can inherit the types, constants, definitions and theorems of other theories as their parents. In the development of the framework, presented in this paper, we make use of the HOL Light theories of Boolean variables, real numbers, transcendental functions and multivariate analysis. In fact, one of the primary motivations of selecting the HOL Light theorem prover for our work was to benefit from these built-in mathematical theories. The proofs in HOL Light are based on the concept of a tactic that breaks goals into simple subgoals. There are many automatic proof procedures and proof assistants available in HOL Light which help the user in directing the proof to the end.

Table 1 provides the mathematical interpretations of some frequently used HOL Light symbols and functions in this paper.

Table 1. HOL Light Symbols and Functions

HOL Symbol	Standard Symbol	Meaning
/\	and	Logical *and*
\/	or	Logical *or*
~	not	Logical *negation*
==>	$\longrightarrow$	Implication
<=>	$=$	Equality in Boolean domain
!x.t	$\forall x.t$	for all $x : t$
λx.t	$\lambda x.t$	Function that maps x to $t(x)$
num	$\{0, 1, 2, \ldots\}$	Positive Integers data type
real	All Real numbers	Real data type
complex	All complex numbers	Complex data type
suc n	$(n + 1)$	Successor of natural number
abs x	$\|x\|$	Absolute function
&a	$\mathbb{N} \rightarrow \mathbb{R}$	Typecasting from Integers to Reals
Cx a	$\mathbb{R} \rightarrow \mathbb{C}$	Typecasting from Reals to Complex
A**B	$[A][B]$	Matrix-Matrix or Matrix-Vector multiplication

4 Proposed Formalization Framework

In this section, we briefly describe the formalization flow to analyze the imaging properties of optical systems using cardinal points. The whole development mainly consists of the following steps (as shown in Figure 3):

Formalization of the Optical System Architecture: The main task of this step is to describe the notion of optical interfaces, free space (both are collectively called an optical component) and optical systems. We also need the formal definitions of some useful functions to check the architectural validity of optical system by ensuring that each component constitutes valid parameters.

Formalization of Light Rays: In the ray optics literature, light rays have been modeled in various ways [6](e.g., sequence of points and wavefront normals etc.). In this step, we formalize the notion of light rays as a sequence of points which is required for the case of matrix modeling of optical systems also called Gaussian Optics. Furthermore, we specify the behavior of these rays when they propagate through optical components and free space.

Verification of Matrix Models: Building upon the above two steps, the next requirement is to verify that any optical component and optical system can

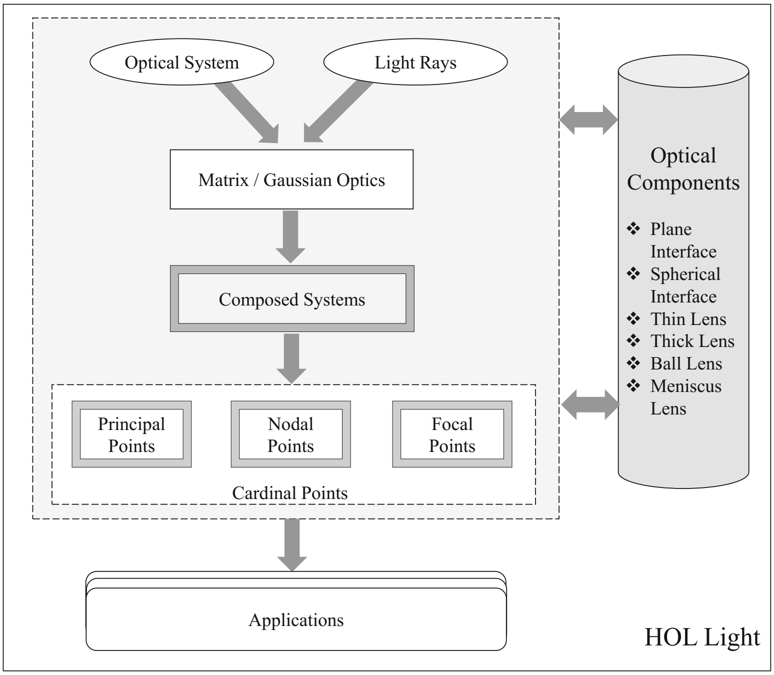


Fig. 3. Framework for the Formalization of the Optical Imaging Systems

be represented as a ray-transfer matrix as given in Equation 1. This step also involves the verification of some helper theorems and lemmas about matrices in HOL Light.

Formalization of Composed Optical Systems: Many optical systems are composed of subsystems and we formalize the notion of composed optical systems in this step. We also need to specify the behavioral characteristics of ray during the propagation through each subsystem.

Formalization of Cardinal Points: We formalize the physical behavior of cardinal points (i.e., principal, nodal and focal points) in object and image space. We then verify the analytical expressions for each of these pair of points for an arbitrary optical system.

Development of a Component Library: It is quite natural to develop a library of the frequently used optical components (e.g., spherical interface, thin lens and thick lens) which mainly consist of the formal modeling, verification of ray-transfer matrix relation and corresponding cardinal points. Finally, the availability of such a library is quite handy to apply our framework to verify the properties of practical optical systems such as visual optical systems (as described in Section 8).

We now present the summary of the first three steps by presenting most important definitions and theorems in the following subsections.

4.1 Modeling of Optical System Structure

Ray optics explains the behavior of light when it passes through a free space and interacts with different interfaces like spherical and plane as shown in Figure 4 (a). We can model free space by a pair of real numbers (n, d), which are essentially the refractive index and the total width, as shown in Figure 1 (a). For the sake of simplicity, we consider only two fundamental interfaces, i.e., plane and spherical which are further categorized as either transmitted or reflected. Furthermore, a spherical interface can be described by its radius of curvature (R). We formalize the above description in HOL Light as follows:

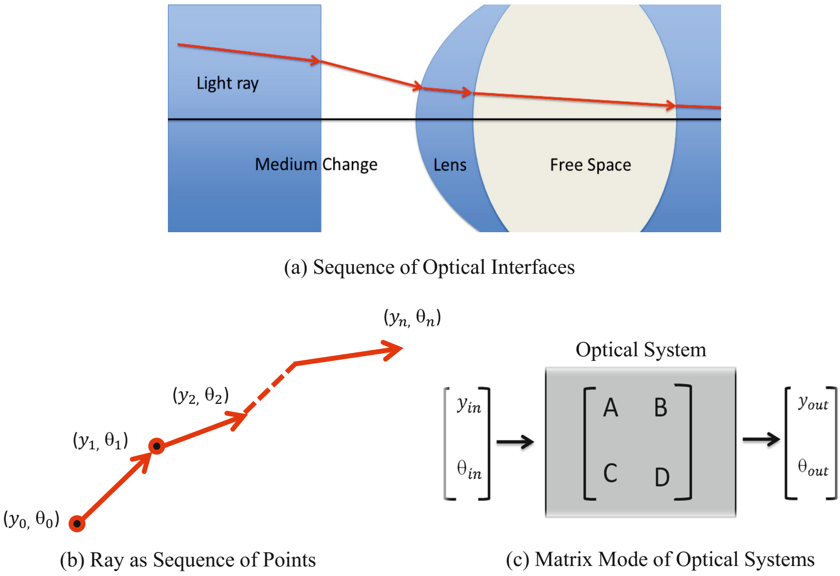


Fig. 4. Schematic Representation of Optical System, Ray and Matrix Model

Definition 1 (Optical Interface and System)

```
define_type "optical_interface = plane | spherical ℝ"
define_type "interface_kind = transmitted | reflected"
new_type_abbrev ("free_space",‘:ℝ × ℝ‘)
```

An optical component is made of a free space (`free_space`) and an optical interface (`optical_interface`) as defined above. Finally, an optical system is a list of optical components followed by a free space. When passing through an interface, the ray is either transmitted or reflected (it is because of the fact that we are only considering sequential ray tracing). In our formalization, this information is also provided in the type definition of optical components, as shown by the use of the type `interface_kind` as follows:

Definition 2 (Optical Interface and System)
```
new_type_abbrev ("optical_component",
            `:free_space × optical_interface × interface_kind`)
new_type_abbrev ("optical_system",
            `:optical_component list × free_space`)
```

Note that this datatype can easily be extended to many other optical components if needed such as new types of lenses or mirrors.

The next step in our formalization is to define some predicates to ensure the validity of free space, optical components and systems. A value of type `free_space` does represent a real space only if the refractive index is greater than zero. We also need to assert the validity of a value of type `optical_interface` by ensuring that the radius of curvature of spherical interfaces is never equal to zero. This yields the following predicates:

Definition 3 (Valid Free Space and Valid Optical Interface)
```
⊢ is_valid_free_space ((n,d):free_space) ⇔ 0 < n
⊢ (is_valid_interface plane ⇔ T) ∧
  (is_valid_interface (spherical R) ⇔ 0 ≠ R)
```

Then, by ensuring that this predicate holds for every component of an optical system, we can characterize valid optical systems as follows:

Definition 4 (Valid Optical Component)
```
⊢ ∀fs i ik. is_valid_optical_component ((fs,i,ik):optical_component)
  ⇔ is_valid_free_space fs ∧ is_valid_interface i
```

Definition 5 (Valid Optical System)
```
⊢ ∀cs fs. is_valid_optical_system ((cs,fs):optical_system) ⇔
  ALL is_valid_optical_component cs ∧ is_valid_free_space fs
```

where `ALL` is a HOL Light library function which checks that a predicate holds for all the elements of a list.

4.2 Modeling of Ray Behavior

We can now formalize the physical behavior of a ray when it passes through an optical system. We only model the points where it hits an optical interface (instead of modeling all the points constituting the ray). So it is sufficient to just provide the distance of all of these hitting points to the axis and the angle taken by the ray at these points as shown in Figure 4 (a) and (b). Consequently, we should have a list of such pairs $(distance, angle)$ for every component of a system. In addition, the same information should be provided for the source of the ray. For the sake of simplicity, we define a type for a pair (*distance*, *angle*) as `ray_at_point`. This yields the following definition:

Definition 6 (Ray)

```
new_type_abbrev ("ray_at_point",`:ℝ×ℝ`)
new_type_abbrev ("ray",
  `:ray_at_point × ray_at_point ×
   (ray_at_point × ray_at_point) list`)
```

The first `ray_at_point` is the pair (*distance*, *angle*) for the source of the ray, the second one is the one after the first free space, and the list of `ray_at_point` pairs represents the same information for the interfaces and free spaces at every hitting point of an optical system.

Once again, we specify what is a valid ray by using some predicates. First of all, we define what is the behavior of a ray when it is traveling through a free space. In paraxial limit, ray travels in a straight line in free space and thus its distance from the optical axis and angle can be related as $y_1 = y_0 + d * \theta_0$ and $\theta_1 = \theta_0$ (as shown in Figure 1), respectively [20]. In order to model this behavior, we require the position and orientation of the ray at the previous and current point of observation, and the free space itself. We encode above information in HOL Light as follows:

Definition 7 (Behavior of a Ray in Free Space)

```
⊢ is_valid_ray_in_free_space
  (y0,θ0) (y1,θ1) ((n,d):free_space) ⇔ y1 = y0 + d * θ0 ∧ θ0 = θ1
```

where (y_0, θ_0), (y_1, θ_1) and (`(n,d):free_space`) represent the ray orientation at previous and current point, and free space, respectively.

Similarly, we define what is the valid behavior of a ray when hitting a particular interface and the propagation in the optical system. Table 2 provides the summary of theses definitions and more implementations details can be found in [23].

Table 2. Some Useful Functions of Ray Optics Formalization

Function	Description
`head_index`	Provides the refractive index of next free space in the optical system
`is_valid_ray_in_free_space`	Provides the mathematical description of ray in free space
`is_valid_ray_at_interface`	Provides the relationship of input and output ray at each interface
`is_valid_ray_in_system`	Ensures that valid behavior of a ray at each interface in the optical system

4.3 Verification of Ray-Transfer Matrices

The main strength of the ray optics is its matrix formulation [26], which provides a convenient way to model all the optical components in the form of a matrix. Indeed, matrix describes a linear relation among input and the output ray as shown in Figure 4 (c). For example, in the case of a free space, the input and output ray parameters are related by two linear equations, i.e., $y_1 = y_0 + d * \theta_0$ and $\theta_1 = \theta_0$, which further can be described in a matrix form as follows:

Theorem 1 (Ray-Transfer-Matrix for Free Space)

$\vdash \forall$n d y_0 θ_0 y_1 θ_1.
  is_valid_free_space (n,d) $\wedge$
  is_valid_ray_in_free_space (y_0,θ_0)
  (y_1,θ_1) (n,d)) $\Longrightarrow$ $\begin{bmatrix} y_1 \\ \theta_1 \end{bmatrix} = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} ** \begin{bmatrix} y_0 \\ \theta_0 \end{bmatrix}$

The first assumption ensures the validity of free space and the second assumption ensures the valid behavior of ray in free space. We use the traditional mathematical notation of matrices for the sake of clarity, whereas we define these matrices using the HOL Light Vectors library. We prove the above theorem using the above mentioned definitions and properties of vectors. Similarly, we prove the ray-transfer matrices of plane and spherical interfaces for the case of transmission and reflection [23].

5 Formalization of Composed Optical Systems

We can trace the input ray R_i through an optical system consisting of n optical components by the composition of ray-transfer matrices of each optical component as described in Equation 1. It is important to note that in this equation, individual matrices of optical components are composed in reverse order. We formalize this fact with the following recursive definition:

Definition 8 (Optical System Model)

$\vdash$ system_composition ([],n,d) $\Leftrightarrow$ free_space_matrix d $\wedge$
  system_composition (CONS ((nt,dt),i,ik) cs,n,d) $\Leftrightarrow$
  (system_composition (cs,n,d) **
  interface_matrix nt (head_index (cs,n,d)) i ik) **
  free_space_matrix dt

General ray-transfer-matrix relation is then given by the following theorem:

Theorem 2 (Ray-Transfer-Matrix for a Single Optical System)

$\vdash \forall$sys ray. is_valid_optical_system sys $\wedge$
  is_valid_ray_in_system ray sys $\Longrightarrow$
  let (y_0,θ_0),(y_1,θ_1),rs = ray in
  let y_n,θ_n = last_ray_at_point ray in
  $\begin{bmatrix} y_n \\ \theta_n \end{bmatrix}$ = system_composition sys ** $\begin{bmatrix} y_0 \\ \theta_0 \end{bmatrix}$

Here, the parameters `sys` and `ray` represent the optical system and the ray respectively. The function `last_ray_at_point` returns the last `ray_at_point` of the ray in the system. Both assumptions in the above theorem ensure the validity of the optical system and the good behavior of the ray in the system. The theorem is easily proved by induction on the length of the system and by using previous results and definitions.

The above described model and corresponding ray-transfer matrix relation only hold for a single optical system consisting of different optical components. Our main requirement is to extend this model for a general system which is composed of n optical subsystems as shown in Figure 5. We formalize the notion of composed optical system as follows:

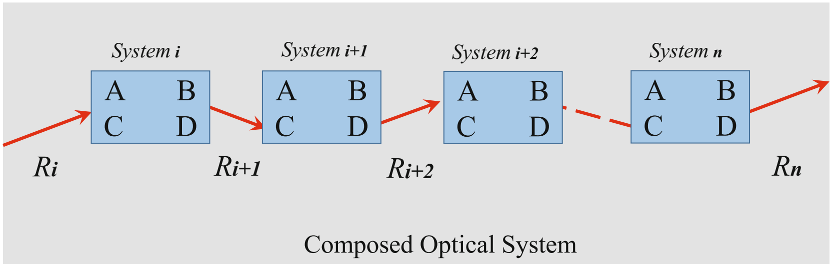


Fig. 5. Ray Propagation through Composed Optical Systems

Definition 9 (Composed Optical System Model)

```
⊢ composed_system [] = I ∧
  composed_system (CONS sys cs) =
  composed_system cs ** system_composition sys
```

where `I` represents the identity matrix and function `composed_system` accepts a list of optical systems `:(optical_system)list` and returns the overall system model by the recursive application of the function `system_composition` (Definition 8). We define the validity of composed optical system by ensuring the validity of each involved optical system as follows:

Definition 10 (Valid Composed Optical System)

```
⊢ ∀(sys:optical_system list). is_valid_composed_system sys ⇔
  ALL is_valid_optical_system sys
```

In order to reason about composed optical systems, we need to give some new definitions about the ray behavior inside a composed optical system. One of the easiest ways is to consider n rays corresponding to n optical systems individually and then make sure that each ray is the same as the one applied at the input. This can be done by ensuring that the starting point of each ray is equal to the ending point of the previous ray as shown in Figure 5. We encode this physical behavior of ray as follows:

Definition 11 (Valid General Ray)
```
⊢is_valid_genray ([]:ray list) ⇔ F ∧
  is_valid_genray (CONS h t) ⇔
  (last_single_ray h = fst_single_ray (HD t) ∧
  is_valid_genray t)
```

where `fst_single_ray`, `last_single_ray` and `HD`, provides the first and last single ray at a point and first element of a list, respectively. On the similar lines, we also specify the behavior of ray when it passes through each optical systems by a function `is_valid_gray_in_system`. Finally, we verify that the ray-transfer-matrix relation holds for composed optical systems which ensures that all valid properties for a single optical system can be generalized to the composed system as well.

Theorem 3 (Ray-Transfer-Matrix for Composed Optical System)
```
⊢ ∀(sys: optical_system list) (ray: ray list).
  is_valid_composed_system sys ∧
  is_valid_gray_in_system ray sys ∧
  is_valid_genray ray ⟹
  let (y0,θ0) = fst_single_ray (HD ray) in
  let (yn,θn) = last_single_ray (LAST ray) in
```
$$\begin{bmatrix} y_n \\ \theta_n \end{bmatrix} = \texttt{composed\_system sys **} \begin{bmatrix} y_0 \\ \theta_0 \end{bmatrix}$$

6 Formalization of Cardinal Points

We consider a general optical imaging system as shown in Figure 6. In this context, the first and the last points of the ray represent the location of object and image. As shown in Figure 6, object (P_0) is located at a distance of d_0 from the optical system and image (P_1) is formed at the distance of d_n. The object and image heights are y_0 and y_n, respectively. The ratio of image height to the object height is called *lateral magnification* which is usually denoted by β. A ray in the object space which intersects the optical axis in the nodal point N at an angle θ intersects the optical axis in the image space in the nodal point N' at the same angle θ'. The ratio of θ and θ' is called *angular magnification*. In our formalization this corresponds to the angle of the first single and and last single ray, respectively. For the sake of generality, we formalize the general notion of optical system as shown in 6, as follows:

Definition 12 (General Optical System Model)
```
⊢ ∀ sys d0 dn ni nt
   gen_optical_system sys d0 dn ni nt ⇔
  [([ ],(ni,d0)); sys; ([ ],(nt,dn))]
```

Here, the overall system consists of 3 sub-systems, i.e., free space with $(\mathtt{n_i}, \mathtt{d_0})$, and general system `sys` and another free space $(\mathtt{n_t}, \mathtt{d_n})$.

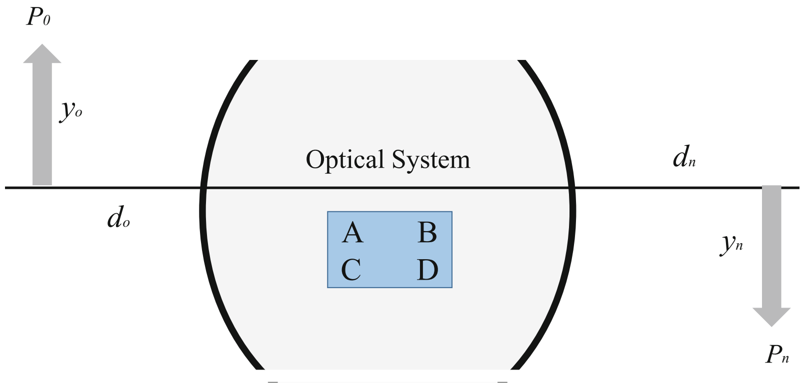


Fig. 6. General Optical System

Our next step is to verify the ray-transfer matrix relation of general optical systems by using Theorem 2, as follows:

Theorem 4 (Matrix for General Optical System)

$\vdash \forall$sys ray $\mathtt{d_0}$ $\mathtt{d_n}$.
  is_valid_optical_system sys $\wedge$ 0 < $\mathtt{n_i}$ $\wedge$ 0 < $\mathtt{n_t}$ $\wedge$
  is_valid_gray_in_system ray sys $\wedge$
  is_valid_genray ray (gen_optical_system sys $\mathtt{d_0}$ $\mathtt{d_n}$ $\mathtt{n_i}$ $\mathtt{n_t}$) $\Longrightarrow$
  let $(\mathtt{y_0}, \theta_0)$ = fst_single_ray (HD ray) in
  let $(\mathtt{y}_n, \theta_n)$ = last_single_ray (LAST ray) in

$$\begin{bmatrix} \mathtt{y_n} \\ \theta_\mathtt{n} \end{bmatrix} = \begin{bmatrix} \mathtt{A} + \mathtt{Cd_n} & (\mathtt{Ad_0} + \mathtt{B} + \mathtt{Cd_0 d_n} + \mathtt{Dd_n}) \\ \mathtt{C} & \mathtt{Cd_0} + \mathtt{D} \end{bmatrix} ** \begin{bmatrix} \mathtt{y_0} \\ \theta_0 \end{bmatrix}$$

Next, we formalize the notion of image and object height, image and object angle, lateral and angular magnification, as follows:

Definition 13 (Lateral and Angular Magnification)

$\vdash \forall$ray. object_height ray = FST (fst_single_ray (HD ray))
$\vdash \forall$ray. image_height ray = FST (last_single_ray (LAST ray))
$\vdash \forall$ray. object_angle ray = SND (fst_single_ray (HD ray))
$\vdash \forall$ray. image_angle ray = SND (last_single_ray (LAST r))

$$\vdash \forall \mathtt{ray.\ lateral\_magnificationray} = \frac{\mathtt{object\_height\ ray}}{\mathtt{image\_height\ ray}}$$

$$\vdash \forall \mathtt{ray.\ angular\_magnificationray} = \frac{\mathtt{object\_angle\ ray}}{\mathtt{image\_angle\ ray}}$$

where `object_height` and `image_height` accept a `ray` and return the lateral distance of image and object from the optical axis, respectively. Similarly, `image_angle` and `object_angle` return the image and object angle, respectively.

The location of all the cardinal points can be found on the optical axis as shown in Figure 2. In case of general optical systems (Figure 6), these can be defined using the distances d_i and d_n, by developing some constraints.

Principal Points: In order to find principal points, the image has to be formed at the same height as of the object in the object space, i.e., the lateral magnification should be one. This means that all the rays, starting from certain height, will have same height regardless of the incident angle. Mathematically this leads to the fact that the second element of 2 × 2 matrix, representing the optical system has to be 0. We package these constraints into the following predicate:

Definition 14 (Principle Points Specification)

```
⊢ ∀(sys: optical_system list).
  principal_points_spec sys ⇔
  (∀ray. is_valid_gray_in_system ray sys ∧ is_valid_genray ray ⟹
  (let M = composed_system sys and
       y_n = image_height ray and
       y_0 = object_height ray in
  y_0 ≠ 0 ∧ M_(2,1) ≠ 0 ⇒
  M_(1,2) = 0 ∧ lateral_magnification ray = 1))
```

The function `principal_points_spec` accepts an arbitrary composed system `sys` and ensures that for any ray the constraints holds as described above. Here, $M_{(i,j)}$ represents the elements of a square matrix M. Now we can define the principle points as the pair of points (dU,dU') which satisfy the above constraints as follows:

Definition 15 (Principle Points of a System)

```
⊢ ∀(sys: optical_system list) dU dU' n_i n_t.
  principal_points (dU,dU') sys ni nt⇔
  principal_points_spec (gen_optical_system sys dU dU' n_i n_t)
```

We used the reasoning support developed in the last section to prove the analytical expressions for the principal points of general optical system described in Figure 6.

Theorem 5 (Principal Points of General System)

```
⊢ ∀n_i n_t sys.
  is_valid_optical_system sys ∧ 0 < n_i ∧ 0 < n_t ∧
  let  M = system_composition sys in
  (principle_points
```

$$((\frac{M_{(2,2)}}{M_{(2,1)}} * (M_{(1,1)} - 1) - M_{(1,2)}), (\frac{1-M_{(1,1)}}{M_{(2,1)}}))\ n_i\ n_t\ sys)$$

Nodal Points: The second cardinal points of an optical system are the nodal points N (in the object space) and N' (in the image space) as shown in Figure 2. A ray in the object space which intersects the optical axis in the nodal point N at an angle θ intersects the optical axis in the image space at the nodal point N' at the same angle θ', which implies that angular magnification should be 1. We encode these constraints as as follows:

Definition 16 (Nodal Points Specification)

```
⊢ ∀(sys: optical_system list).
  nodal_points_spec sys ⇔
  (∀ray. is_valid_gray_in_system ray sys ∧ is_valid_genray ray ⟹
  (let M = composed_system sys and
       y_n = image_height ray and
       y_n = image_height ray and
       θ_0 = object_angle ray and
       θ_n = image_angle ray in
  y_0 = 0 ∧ y_n = 0 ∧ θ_0≠ 0∧ M_(2,1) ≠ 0 ⇒
  M_(1,2) = 0 ∧ angular_magnification ray = 1))
```

The function `nodal_points_spec` accepts an arbitrary composed system `sys` and ensures that for any ray the constraints holds as described above. Consequently, we can define the nodal points as the pair of points (dN,dN') which satisfy the above constraints as follows:

Definition 17 (Nodal Points of a System)

```
⊢ ∀(sys: optical_system list) dU dU' n_i n_t.
  nodal_points (dN,dN') sys ni nt⇔
  nodal_points_spec (gen_optical_system sys dU dU' n_i n_t)
```

The corresponding analytical expressions for the Nodal points of general optical system described are proved in following theorem.

Theorem 6 (Nodal Points of General System)

```
⊢ ∀n_i n_t sys.
  is_valid_optical_system sys ∧ 0 < n_i ∧ 0 < n_t ∧
  let  M = system_composition sys in
  (nodal_points
  (((1−M_(2,2))/M_(2,1)),(M_(1,1)/M_(2,1) ∗ (M_(2,2) − 1) − M_(1,2))) n_i n_t sys)
```

Focal Points: The focal points F (in the object space) and F' (in the image space), have two properties: A ray starting from the focus F in the object space is transformed into a ray which is parallel to the optical axis in the image space. Similarly, a ray which is parallel to the optical axis in the object space intersects the focus F' in the image space. We define the following predicate using the above description:

Definition 18 (Focal Points Specification)

```
⊢ ∀(sys: optical_system list).
  focal_points_spec sys ⇔
  (∀ray. is_valid_gray_in_system ray sys ∧ is_valid_genray ray ⟹
  (let M = composed_system sys and
       y_n = image_height ray and
       y_n = image_height ray and
       θ_0 = object_angle ray and
       θ_n = image_angle ray in
  M_(2,1) ≠ 0 ⇒
  (θ_n = 0 ∧ y_0 = 0 ⇒ M_(1,1) ≠ 0) ∧
  (θ_0 = 0 ∧ y_n = 0 ⇒ M_(2,2) ≠ 0)
```

Finally, we can define the focal points (dF,dF') as follows:

Definition 19 (Focal Points of a System)

```
⊢ ∀(sys: optical_system list) dU dU' n_i n_t.
  focal_points (dF,dF') sys ni nt⇔
  focal_points_spec (gen_optical_system sys dU dU' n_i n_t)
```

We also verify the corresponding analytical expressions for the focal points points in the following theorem.

Theorem 7 (Focal Points of General System)

```
⊢ ∀n_i n_t sys.
  is_valid_optical_system sys ∧ 0 < n_i ∧ 0 < n_t ∧
  let  M = system_composition sys in
  (focal_points ((-M_(2,2) / M_(2,1)),(-M_(1,1) / M_(2,1))) n_i n_t sys)
```

This completes the formalization of cardinal points of the optical systems. Theorems 5,6 and 7 are powerful results as they simplify the calculation of cardinal points to just finding an equivalent matrix of the given optical system.

7 Cardinal Points of Frequently Used Optical Components

In this section, we present the summary of the formal verification of the cardinal points of widely used optical components. Generally, lenses are characterized by their refractive indices, thickness and radius of curvature in case of spherical interface. Some of the components are shown in Figure 7, i.e., refracting spherical interface, thick lens, ball lens and plano convex lens. Note that all of these components are composed of two kinds of interfaces, i.e., plane or spherical and free spaces of different refractive indices and widths. We use our developed infrastructure to formalize these components and verify the ray-transfer-matrix relation for each model. Consequently, we can easily derive the cardinal points using already verified theorems. For the sake of conciseness, we only present the formalization of thick lens and the verification of its principal points. A thick lens is a composition of two spherical interfaces separated by a distance d as shown in Figure 7 (b). We formalize thick lenses as follows:

Definition 20 (Thick Lens)

```
⊢ ∀R1 R2 n1 n2 d. thick_lens R1 R2 n1 n2 d =
  ([(n1,0),spherical R1,transmitted;
  (n2,d),spherical R2,transmitted],(n1,0))
```

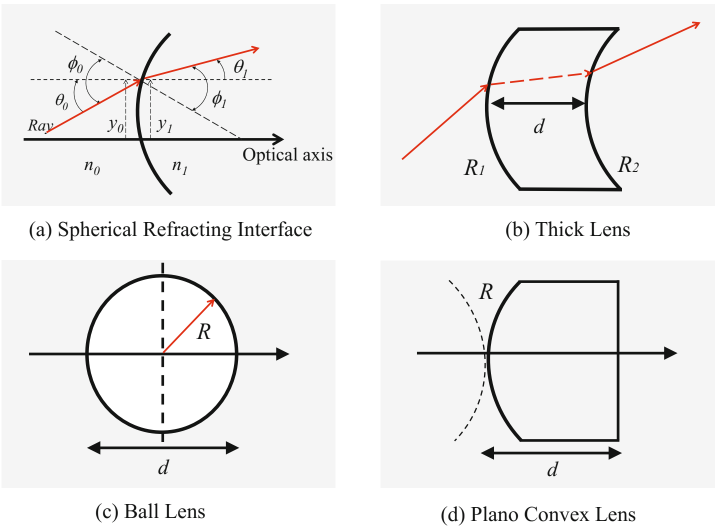
where n_1 represents the refractive index before and after the first and the second interface, respectively. Whereas n_2 represents the refractive index between the two spherical interfaces which have the radius of curvatures R_1 and R_2, respectively.

We then verify the general expression for the principal points of a thick lens in the following theorem.

Theorem 8 (Principal Points of Thick Lens)

```
⊢∀R1 R2 n0 n1 d. R1 ≠ 0 ∧ R2 ≠ 0 ∧ 0 < n1 ∧ 0 < n2 ∧
  (d * (n1 - n2) ≠ -n2 * (R1 - R1)) ⟹
  (let dU = (n * d * R1) / (n2 * (R2 - R1) + (n2 - n1) * d) and
       dU' = -(n * d * R2) / (n2 * (R2 - R1) + (n2 - n1) * d) in
  principal_points (dU,dU') (thick_lens R1 R2 n1 n2 d) n1 n1)
```

Here, the first four assumptions are required to verify the validity of the thick lens structure and the last assumption specifies the condition about thick lens parameters which is required to verify the principal points dU and dU'. Similarly, we verify the principal points for other optical component as given in Table 3. Moreover, we also formalize some other optical components such as thin lens and parallel plate where complete details can be found in the source code [24].



(a) Spherical Refracting Interface

(b) Thick Lens

(c) Ball Lens

(d) Plano Convex Lens

Fig. 7. Frequently used Optical Components [26]

Table 3. Principal Points of Some Optical Components

Optical Component	**Principal Points**
Spherical Interface (transmitted)	$dU = 0 \wedge dU' = 0$
Spherical Interface (Reflected)	$dU = 0 \wedge dU' = 0$
Ball Lens	$dU = -R \wedge dU' = -R$
Meniscus Lens	$dU = \frac{R}{n_L - 1} \wedge dU' = -\frac{R}{n_L - 1}$
Plano Convex Lens	$dU = 0 \wedge dU' = -\frac{d}{n_L}$

This completes the formal verification of the cardinal points of the optical imaging systems which to the best of our knowledge is done for the first time using theorem proving. Due to the formal nature of the model and the inherent soundness of higher-order logic theorem proving, we have been able to verify generic results such as Theorems 5,6 and 7. This improved accuracy comes at the cost of the time and efforts spent, while formalizing the underlying theory of geometrical optics and composed optical systems. Interestingly, the availability of such a formalized infrastructure significantly reduces the time required to analyze the cardinal points of the frequently used optical components. Another contribution of our work is to bring out all the hidden assumptions about the physical models of lenses and mirrors which otherwise are not mentioned in the optics literature (e.g., [26]). Moreover, we automatized parts of the verification task by introducing new tactics. Some of these tactics are specialized to verify (or simplify) the proofs related to our formalization of ray optics (e.g., `VALID_OPTICAL_SYSTEM_TAC` [24]). However, some tactics are general and can be used in different verification tasks involving matrix/vector operations. An example of such tactic is `COMMON_TAC`, which allows us to verify the ray-transfer matrices in our development.

8 Formal Analysis of Visual Optical System for an Eye

Human eye is a complex optical system which processes light rays through different biological layers such as cornea, iris and crystalline lens which is located directly behind the pupil. There are different eye diseases some of them are age related and others are caused due to the malfunctioning of some tissues inside the eye. Myopia (or near-sightedness) is a commonly found eye disease which is caused due to the wrong focus of the incoming light inside the eye. In general, myopia is considered as a significant issue due to its high prevalence and the risk for vision-threatening conditions as described in the guidelines by American Optometric Association [2]. The most commonly used method to avoid this problem is by the use of corrective lenses or eye surgery [2]. Mathematically, different conditions for myopia can be analyzed using geometrical optics and cardinal points [10]. We consider the general description of the visual optical system of eye as shown in Figure 8. The visual optical system of an eye is described by S and an optical device is represented by S_D. The parameter S_G is a homogeneous gap between S_D and the eye, S_E is the combination of S_D and S_G. Similarly, S_C is the combination of S_E and S. The points Q_0 and Q_1 are the incident and emergent special points of S and Q_{C0} and Q_{C1} are the corresponding cardinal points (can be either principal, nodal and focal points) of S_C. When, we place S_D in front of the eye, it causes Q_0 to be displaced by Δ_{QC0} and Q_1 to be displaced by Δ_{QC1}. In this design, the entrance plane T_0 is located immediately anterior to the first surface of the tear layer on the cornea and the exit plane T_1 is located immediately anterior to the retina of the eye. Our main goal is to formally derive the cardinal points for this systems description. We proceed by the formal model which consists of three main subsystems:

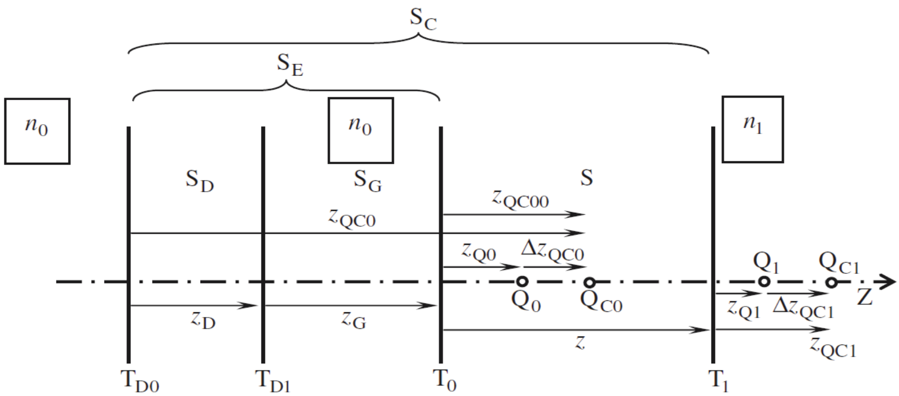


Fig. 8. Visual Optical System for an Eye [10]

- The visual optical system of the eye S.
- Homogeneous distance S_G: it can be modeled using a free space of width z_G.
- Any corrective optical device S_D: it can be a contact lens or some surgical equipment.

The corresponding HOL Light definition is as follows:

Definition 21 (Model of the Optical Corrective Setup for Myopia)

$\vdash \forall$ `system_eye` $\mathtt{z_G}$ `device`

`EYE_corrective_sys system_eye` $\mathtt{z_G}$`device` $\Leftrightarrow$

$[\mathtt{system\_eye}; ([\ ], (\mathtt{1}, \mathtt{z_G})); \mathtt{device}]$

We now derive the general expressions for the cardinal points as follows:

Theorem 9 (Cardinal Points of General System)
$\vdash \forall$ system_eye z_G device.
is_valid_optical_system system_eye $\wedge$
is_valid_optical_system device $\Longrightarrow$
let $\begin{bmatrix} A_c & B_c \\ C_c & D_d \end{bmatrix}$ =
composed_system (EYE_corrective_sys system_eye z_G device) in
principle_points $((\frac{D_c}{C_c} * (A_c - 1) - B_c), (\frac{1-A_c}{C_c}))$
n_i n_t EYE_corrective_sys system_eye z_G device $\wedge$
nodal_points $((\frac{1-A_c}{C_c}), (\frac{A_c}{C_c} * (D_c - 1) - B_c))$
n_i n_t EYE_corrective_sys system_eye z_G device $\wedge$
focal_points $((-\frac{D_c}{C_c}), (-\frac{A_c}{C_c}))$
n_i n_t EYE_corrective_sys system_eye z_G device

Given the structure of the corrective device, we can easily find the location of Q_{C0} and Q_{C1}, i.e., cardinal points which help to estimate the shifts in the cardinal points of the visual system of eye. Furthermore different decisions about the diagnoses of a disease can be made based on the equivalent composed system. For example, the element A_c is the direct measure of the myopia of the eye, i.e., the eye is myopic, emmetropic or hyperopic if A_c is negative, zero or positive, respectively [10]. All the results are derived in general form which can be directly used for particular corrective devices and the parameters of eye.

9 Conclusion

In this paper, we reported a new application of formal methods to verify the imaging properties of optical systems. In particular, we provided a brief introduction of the current state-of-the-art and highlighted their limitations. We formalized the notion of composed optical systems and verify that composed systems inherit the same linear algebraic properties as for the case of a single optical system. Consequently, we formalized the notion of cardinal points of an optical systems along with the verification of the generic expressions for the case of an arbitrary optical system. Finally, we presented the formal analysis of a vision corrective biomedical device to analyze the myopia. The main challenge of the reported work is its interdisciplinary nature due to the involvement of optical physics, mathematics and interactive theorem proving. Based on our experience, we believe that there is a lot of potential to apply formal methods to verify biomedical systems in general and surgical devices in particular. One obvious hurdle is the gap among the theorem proving and other physical sciences such as biology, optics and fluid dynamics.

Our future work is to formalize and verify the correctness and soundness of the ray tracing algorithm [26], which is included in almost all optical systems design tools. Other future directions include the application of our work in the analysis of ophthalmic devices which are both cost and safety critical.

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