Higher-Order Logic Formalization of Geometrical Optics

Umair Siddique, Vincent Aravantinos and Sofiène Tahar

Department of Electrical and Computer Engineering,
Concordia University, Montreal, Canada
{muh_sidd,vincent,tahar}@ece.concordia.ca

Abstract. Geometrical optics, in which light is characterized as rays, provides an efficient and scalable formalism for the modeling and analysis of optical and laser systems. The main applications of geometrical optics are in stability analysis of optical resonators, laser mode locking and micro opto-electro-mechanical systems. Traditionally, the analysis of such applications has been carried out by informal techniques like paper-and-pencil proof methods, simulation and computer algebra systems. These traditional techniques cannot provide 100% accurate results and thus cannot be recommended for safety-critical applications, such as corneal surgery, process industry and inertial confinement fusion. On the other hand, higher-order logic theorem proving does not exhibit the above limitations, thus we propose a higher-order logic formalization of geometrical optics. Our formalization is mainly based on existing higher-order logic theories of geometry and multivariate analysis in the HOL Light theorem prover. In order to demonstrate the practical effectiveness of our formalization, we present the formal stability analysis of optical and laser resonators.

1 Introduction

Different characterizations of light lead to different fields of optics such as quantum optics, electromagnetic optics, wave optics and geometrical optics. The latter describes light as rays which obey geometrical rules. The theory of geometrical optics can be applied for the modeling and analysis of physical objects with dimensions greater than the wavelength of light. Geometrical optics is based on a set of postulates which are used to derive the rules for the propagation of light through an optical medium. These postulates can be summed up as follows: Light travels in the form of rays emitted by a source; an optical medium is characterized by its refractive index; light rays follow Fermat's principle of least time [17].

Optical components, such as thin lenses, thick lenses and prisms are usually centered about an optical axis, around which rays travel at small inclinations (angle with the optical axis). Such rays are called *paraxial rays* and this assumption provides the basis of *paraxial optics* which is the simplest framework of geometrical optics. The paraxial approximation explains how light propagates through a series of optical components and provides diffraction-free description of complex optical systems. The change in the position and inclination of a paraxial ray as it travels through an optical system can be efficiently described by the use of a matrix algebra [10]. This matrix formalism (called *ray-transfer matrices*) of geometrical optics provides accurate, scalable and systematic analysis of real-world complex optical and laser systems. This fact has led to the widespread usage of ray-transfer matrices in the modeling and analysis of critical physical systems. Typical applications of ray-transfer matrices include analysis of a laser beam propagation through some optical setup [10], the stability analysis of laser or optical resonators [12], laser mode-locking, optical pulse transmission [14] and analysis of micro opto-electro-mechanical systems (MOEMS) [20]. Another promising feature of the matrix formalism of geometrical optics is the prediction of design parameters for physical experiments, e.g., recent dispersion-managed soliton transmission experiment [13] and invention of the first single-cell biological lasers [3].

Traditionally, the analysis of geometrical optics based models has been done using paper-and-pencil proof methods [10, 14, 13]. However, considering the complexity of present age optical and laser systems, such an analysis is very difficult if not impossible, and thus quite error-prone. Many examples of erroneous paper-and-pencil based proofs are available in the open literature, a recent one can be found in [2] and its identification and correction is reported in [15]. One of the most commonly used computer-based analysis techniques for geometrical optics based models is numerical computation of complex ray-transfer matrices [19, 11]. Optical and laser systems involve complex and vector analysis and thus numerical computations cannot provide perfectly accurate results due to the inherent incomplete nature of the underlying numerical algorithms. Another alternative is computer algebra systems [16], which are very efficient for computing mathematical solutions symbolically, but are not 100% reliable due to their inability to deal with side conditions [5]. Another source of inaccuracy in computer algebra systems is the presence of unverified huge symbolic manipulation algorithms in their core, which are quite likely to contain bugs. Thus, these traditional techniques should not be relied upon for the analysis of critical laser and optical systems (e.g., corneal surgery [9]), where inaccuracies in the analysis may even result in the loss of human lives.

In the past few years, higher-order logic theorem proving has been successfully used for the precise analysis of a few continuous physical systems [18, 8]. Developing a higher-order logic model for a physical system and analyzing this model formally is a very challenging task since it requires both a good mathematical and physical knowledge. However, it provides an effective way for identifying critical design errors that are often ignored by traditional analysis techniques like simulation and computer algebra systems. We believe that higher-order logic theorem proving [4] offers a promising solution for conducting formal analysis of such critical optical and laser systems. Most of the classical mathematical theories behind geometrical optics, such as Euclidean spaces, multivariate analysis and complex numbers, have been formalized in the HOL Light theorem prover

[6, 7]. To the best of our knowledge, the reported formalization of geometrical optics is the first of its kind.

2 Geometrical Optics

When a ray passes through optical components, it undergoes *translation* or *refraction*. In translation, the ray simply travels in a straight line from one component to the next and we only need to know the thickness of the translation. On the other hand, refraction takes place at the boundary of two regions with different refractive indices and the ray obeys the law of refraction, i.e., the angle of refraction relates to the angle of incidence by the relation $n_0 \sin(\phi_0) = n_1 \sin(\phi_1)$, called *Snell's law* [17], where n_0, n_1 are the refractive indices of both regions and ϕ_0, ϕ_1 are the angles of the incident and refracted rays, respectively, with the normal to the surface. In order to model refraction, we thus need the normal to the refracting surface and the refractive indices of both regions.

In order to introduce the matrix formalism of geometrical optics, we consider the propagation of a ray through a spherical interface with radius of curvature R between two mediums of refractive indices n_0 and n_1, as shown in Figure 1. Our goal is to express the relationship between the incident and refracted rays. The trajectory of a ray as it passes through various optical components can be specified by two parameters: its distance from the optical axis and its angle with the optical axis. Here, the distances of the incident and refracted rays are r_1 and r_0, respectively, and $r_1 = r_0$ because the thickness of the surface is assumed to be very small. Here, ϕ_0 and ϕ_1 are the angles of the incident and refracted rays with the normal to the spherical surface, respectively. On the other hand, θ_0 and θ_1 are the angles of the incident and refracted rays with the optical axis.

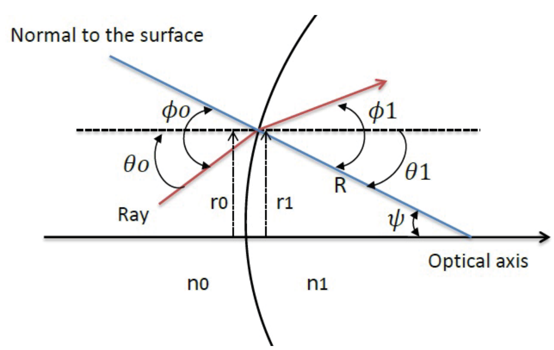

Fig. 1. Spherical interface

Applying Snell's law at the interface, we have $n_0 \sin(\phi_0) = n_1 \sin(\phi_1)$, which, in the context of paraxial approximation, reduces to the form $n_0\phi_0 = n_1\phi_1$ since $\sin(\phi) \simeq \phi$ if ϕ is small. We also have $\theta_0 = \phi_0 - \psi$ and $\theta_1 = \phi_1 - \psi$, where ψ

is the angle between the surface normal and the optical axis. Since $\sin(\psi) = \frac{r_0}{R}$, then $\psi = \frac{r_0}{R}$ by paraxial approximation. We can deduce that:

$$\theta_1 = \left(\frac{n_0 - n_1}{n_1 R}\right) r_0 + \left(\frac{n_0}{n_1}\right) \theta_0 \tag{1}$$

So, for a spherical surface, we can relate the refracted ray with the incident ray by a matrix relationship using equation (1) as follows:

$$\begin{bmatrix} r_1 \\ \theta_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{n_0 - n_1}{n_1 R} & \frac{n_0}{n_1} \end{bmatrix} \begin{bmatrix} r_0 \\ \theta_0 \end{bmatrix}$$

Thus the propagation of a ray through a spherical interface can be described by a 2 × 2 matrix generally called, in the literature, *ABCD matrix*. This can be generalized to many optical components [17] as follows:

$$\begin{bmatrix} r_1 \\ \theta_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} r_0 \\ \theta_0 \end{bmatrix}$$

where matrix elements are either real in the case of spatial domain analysis or complex in the case of time domain analysis [14]. If we have an optical system consisting of k optical components, then we can trace the input ray R_i through all optical components using composition of matrices of each optical component as follows:

$$R_o = (M_k.M_{k-1}....M_1).R_i \tag{2}$$

Simply, we can write $R_o = M_s R_i$ where $M_s = \prod_{i=k}^{1} M_i$. Here, R_o is the output ray and R_i is the input ray. In the next section, we present a brief overview of our higher-order logic formalization of geometrical optics.

3 Formalization of Geometrical Optics

The formalization is two-fold: first, we model the geometry and physical parameters of an optical system; second, we model the physical behavior of a ray when it goes through an optical interface. Afterwards, we will be able to derive the ray-transfer matrices of the optical components, as explained in Section 2.

An optical system is a sequence of optical interfaces, which are defined by an inductive data type enumerating their different kinds and their corresponding parameters:

Definition 1 (Optical Interface and System).

```
define_type "optical_interface = plane | spherical real"
define_type "interface_kind = transmitted | reflected"
new_type_abbrev("free_space",`:real # real`)
new_type_abbrev("optical_system",`:(free_space # optical_interface #
                                   interface_kind) list # free_space`)
```

An optical system is made of a list of free spaces which are formalized by pairs of real numbers representing the refractive index and width of free space, and a list of optical interfaces along with their types describing the system itself. Optical interfaces themselves are of two kinds: plane or spherical interfaces, yielding the corresponding constructors as shown in Figure 2. Both plane and spherical interface are of two types, i.e., transmitted and reflected which characterize their behavior either the incident ray will pass through or reflects back. A spherical interface takes a real number representing its radius of curvature. Note that this datatype can easily be extended to many other optical components if needed. Here, we call (`free_space`,`optical_interface`,`interface_kind`) an *optical component*.

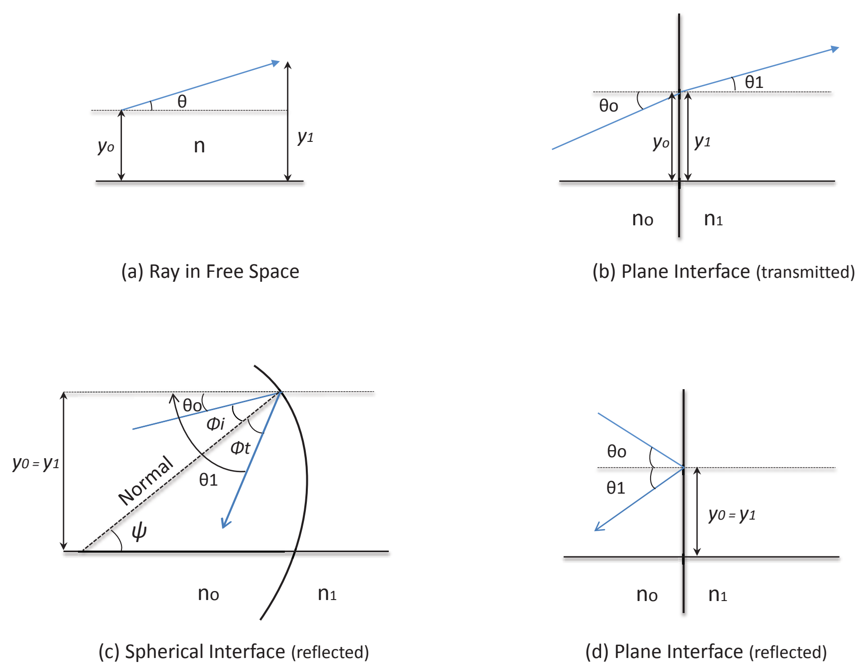

Fig. 2. Behavior of ray at different interfaces

A value of the type `free_space` does represent a real space only if the refractive index is greater than zero. In addition, in order to have a fixed order in the representation of an optical system, we impose that the distance of an optical interface relative to the previous interface is greater or equal to zero. We also need to assert the validity of a value of type `optical_interface` by ensuring that the radius of curvature of spherical interfaces is never equal to zero. This yields the following predicates:

Definition 2 (Valid Free Space and Valid Optical Component).

```
⊢ is_valid_free_space ((n,d):free_space) ⇔ 0 < n ∧ 0 ≤ d
⊢ (is_valid_interface plane ⇔ T) ∧
  (is_valid_interface (spherical_interface R) ⇔ 0 <> R)
```

Then, by ensuring that this predicate holds for every component of an optical system, we can characterize valid optical systems as follows:

Definition 3 (Valid Optical System).

```
⊢ ∀os fs. is_valid_optical_system ((cs,fs):optical_system) ⇔
  is_valid_free_space fs ∧ ALL (λ(fs,i,ik).is_valid_free_space fs ∧
  is_valid_interface i) cs
```

where `ALL` is a HOL Light library function which checks that a predicate holds for all the elements of a list. We conclude our formalization of an optical system by defining the following helper function to retrieve the refractive index of the first free space in an optical system:

Definition 4 (Refractive Index of First Free Space).

```
⊢ (head_index ([],(n,d)) = n) ∧
  (head_index (CONS ((n,d),i) cs, (nt,dt)) = n)
```

We can now formalize the physical behavior of a ray when it passes through an optical system. We only model the points where a ray hits an optical interface (instead of all the points constituting the ray). So it is sufficient to just provide the distance of the hitting point to the axis and the angle taken by the ray at that point. Consequently, we should have a list of such pairs $(distance, angle)$ for every component of a system. In addition, the same information should be provided for the source of the ray. For the sake of simplicity, we define a type for a pair $(distance, angle)$ as `ray_at_point`. This yields the following definition:

Definition 5 (Ray).

```
new_type_abbrev ("ray_at_point", ‘:real # real‘)
new_type_abbrev ("ray", ‘:ray_at_point # ray_at_point #
                         (ray_at_point # ray_at_point) list‘)
```

The first `ray_at_point` is the pair $(distance, angle)$ for the source of the ray, the second one is the one after the first free space, and the list of `ray_at_point` represents the same information for all hitting points of an optical system. The reason behind the list of `ray_at_point` is because an optical system is modeled as a list of free space and interface.

Once again, we specify what is a valid ray by using some predicates. First of all, we define what is the behavior of a ray when it is traveling through a free space. This requires the position and orientation of the ray at the previous and current point of observation, and the free space itself. This is shown in Figure 2(a).

Definition 6 (Behavior of a Ray in Free Space).

```
⊢ is_valid_ray_in_free_space (y₀,θ₀) (y₁,θ₁) ((n,d):free_space) ⇔
                              y₁ = y₀ + d * θ₀ ∧ θ₀ = θ₁
```

Next, we define what is the valid behavior of a ray when hitting a particular interface. This requires the position and orientation of the ray at the previous and current interface, and the refractive index before and after the component. Then the predicate is defined by case analysis on the interface and its type as follows:

Definition 7 (Behavior of a Ray at Given Interface).

```
⊢ (is_valid_ray_at_interface (y₀,θ₀) (y₁,θ₁) n₀ n₁ plane transmitted
  ⇔ y₁ = y₀ ∧ n₀ * θ₀ = n₁ * θ₁) ∧
  (is_valid_ray_at_interface (y₀,θ₀) (y1,θ₁) n₀ n₁ (spherical R)
   transmitted ⇔ let φᵢ= θ₀ + y₁/R and φₜ = θ₁ + y₁/R in
   y₁ = y₀ ∧ n₀ * φᵢ = n₁ * φₜ) ∧
  (is_valid_ray_at_interface (y₀,θ₀) (y₁,θ₁) n₀ n₁ plane reflected
   ⇔ y₁ = y₀ ∧ n₀ * θ₀ = n₀ * θ₁) ∧
  (is_valid_ray_at_interface (y₀,θ₀) (y₁,θ₁) n₀ n₁ (spherical R)
   reflected ⇔ let φᵢ = y1/R - θ₀ in y₁ = y₀ ∧ θ₁ = -(θ₀ + 2 * φᵢ))
```

The above definition states some basic geometrical facts about the distance to the axis, and applies Snell's law to the orientation of the ray as shown in Figures 1 and 2. Note that, both to compute the distance and to apply Snell's law, we assumed the paraxial approximation in order to turn $\sin(\theta)$ into θ. Finally, we can recursively apply these predicates to all the componets of a system as follows:

Definition 8 (Behavior of a Ray in an Optical System).

```
⊢ ∀ sr₁ sr₂ h h' fs cs rs i ik y₀ θ₀ y₁ θ₁ y₂ θ₂ y₃ θ₃ n d n' d'.
  (is_valid_ray_in_system (sr₁,sr₂,[]) (CONS h cs,fs) ⇔ F) ∧
  (is_valid_ray_in_system (sr₁,sr₂,CONS h' rs) ([],fs) ⇔F)∧
  (is_valid_ray_in_system ((y₀,θ₀),(y₁,θ₁),[]) ([],n,d) ⇔
   is_valid_ray_in_free_space (y₀,θ₀) (y₁,θ₁) (n,d)) ∧
  (is_valid_ray_in_system ((y₀,θ₀),(y₁,θ₁),
   CONS ((y₂,θ₂),y₃,θ₃) rs) (CONS ((n',d'),i,ik) cs,n,d) ⇔
  (is_valid_ray_in_free_space (y₀,θ₀) (y₁,θ₁) (n',d') ∧
   is_valid_ray_at_interface (y₁,θ₁) (y₂,θ₂) n'
   (head_index (cs,n,d)) i ik)) ∧
  (is_valid_ray_in_system ((y₂,θ₂),(y₃,θ₃),rs) (cs,n,d))
```

The behavior of a ray going through a series of optical components is thus completely defined. Using this formalization, we verify the ray-transfer matrices as presented in Section 2. In order to facilitate formal reasoning, we define the following matrix relations for free spaces and interfaces.

Definition 9 (Free Space Matrix).

$$\vdash \forall\ \mathtt{d.\ free\_space\_matrix\ d} \Leftrightarrow \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}$$

Definition 10 (Interface Matrix).
$\vdash \forall$ `n`$_0$ `n`$_1$ `R.`

`interface_matrix` n_0 n_1 `plane transmitted` $\Leftrightarrow \begin{bmatrix} 1 & 0 \\ 0 & \frac{n_0}{n_1} \end{bmatrix} \wedge$

`interface_matrix` n_0 n_1 `(spherical R) transmitted` $\Leftrightarrow \begin{bmatrix} 1 & 0 \\ \frac{n_0 - n_1}{n_0 * R} & \frac{n_0}{n_1} \end{bmatrix} \wedge$

`interface_matrix` n_0 n_1 `plane reflected` $\Leftrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \wedge$

`interface_matrix` n_0 n_1 `(spherical R) reflected` $\begin{bmatrix} 1 & 0 \\ \frac{-2}{R} & 1 \end{bmatrix}$

In the above definition, n_0 and n_1 represent the refractive indices before and after an optical interface. We use the traditional mathematical notation of matrices for the sake of clarity, whereas we define these matrices using the HOL Light Vectors library. For example, a simple `2 x 2` matrix can be defined as follows:

Definition 11 (Matrix in HOL Light).
$\vdash \forall$ `A B C D. matrix_abcd A B C D` $\Leftrightarrow$ `vector[vector[A;B];vector[C;D]]`

Next, we verify the ray-transfer-matrix relation for free space:

Theorem 1 (Ray-Transfer-Matrix for Free Space).
$\vdash \forall$ `n d` y_0 θ_0 y_1 θ_1`. is_valid_free_space (n,d)` $\wedge$
`is_valid_ray_in_free_space` (y_0,θ_0) (y_1,θ_1) `(n,d))` $\Longrightarrow$
$\begin{bmatrix} y_1 \\ \theta_1 \end{bmatrix}$ `= free_space_matrix d *` $\begin{bmatrix} y_0 \\ \theta_0 \end{bmatrix}$

Here, the first assumption ensures the validity of free space and the second assumption ensures the valid behavior of ray in free space. The proof of this theorem requires some properties of vectors and matrices along with some arithmetic reasoning. Next, we verify an important theorem describing general ray-transfer-matrix relation for any interface as follows:

Theorem 2 (Ray-Transfer-Matrix any Interface).
$\vdash \forall$ n_0 n_1 y_0 θ_0 y_1 θ_1 `i ik. is_valid_interface i` $\wedge$
`is_valid_ray_at_interface` (y_0,θ_0) (y_1,θ_1) n_0 n_1 `i ik` $\wedge$
$0 < n_0 \wedge 0 < n_1 \Longrightarrow \begin{bmatrix} y_1 \\ \theta_1 \end{bmatrix}$ `= interface_matrix` n_0 n_1 `i ik *` $\begin{bmatrix} y_0 \\ \theta_0 \end{bmatrix}$

In the above theorem, both assumptions ensure the validity of interface and behavior of ray at the interface, respectively. This theorem is easily proved by case splitting on `i` and `ik`.

Now, equipped with the above theorem, the next step is to formally verify the ray-transfer-matrix relation for an optical system as given in Equation (2). It is important to note that in Equation (2), individual matrices of optical components are composed in reverse order. We formalize this fact with the following recursive definition:

Definition 12 (System Composition).

⊢ system_composition ([],n,d) ⇔ free_space_matrix d ∧
system_composition (CONS ((nt,dt),i,ik) cs,n,d) ⇔
(system_composition (cs,n,d) *
interface_matrix nt (head_index (cs,n,d)) i ik) *
free_space_matrix dt

The general ray-transfer-matrix relation is then given by the following theorem:

Theorem 3 (Ray-Transfer-Matrix for Optical System).

⊢ ∀ sys ray. is_valid_optical_system sys ∧
is_valid_ray_in_system ray sys ⟹
let $(y_0,\theta_0),(y_1,\theta_1)$,rs = ray in
let y_n,θ_n = last_ray_at_point ray in

$$\begin{bmatrix} y_n \\ \theta_n \end{bmatrix} = \texttt{system\_composition sys} * \begin{bmatrix} y_0 \\ \theta_0 \end{bmatrix}$$

Here, the parameters sys and ray represent the optical system and the ray respectively. The function last_ray_at_point returns the last ray_at_point in system. Both assumptions in the above theorem ensure the validity of the optical system and the good behavior of the ray in the system. The theorem is easily proved by induction on the length of the system and by using previous results and definitions.

This concludes our formalization of geometrical optics and verification of important properties of optical components and optical systems. The formal verification of the above theorems not only ensures the effectiveness of our formalization but also shows the correctness of our formal definitions related to optical systems. Now, we present the formal verification of the ray-transfer matrix relationship of Thin Lenses [17], which is one of the most widely used components in optical and laser systems.

Generally, lenses are determined by their refractive indices and thickness. In thin lens approximation, a lens is considered as the composition of two transmitted spherical interfaces and any variation of ray parameters (position y and orientation θ) is neglected between both interfaces, as shown in Figure 3. So, a thin lens is the composition of two spherical interfaces with a null width free space in between. Now, we present the formal verification of the thin lens matrix.

Theorem 4 (Thin Lens Matrix).

⊢ ∀ R_1 R_2 n_0 n_1. R_1 <> 0 ∧ R_2 <> 0 ∧ 0 < n_1 ∧ 0 < n_2 ∧ ⟹
system_composition ([(n_1,0),spherical R_1,transmitted ; (n_2,0),
spherical R_2,transmitted], n_1,0) ⇔

$$\begin{bmatrix} 1 & 0 \\ \frac{n_2-n_1}{n_1}\left(\frac{1}{R_2}-\frac{1}{R_1}\right) & 1 \end{bmatrix}$$

In the next section, we sketch the formal stability analysis of an optical resonator using our formalization.

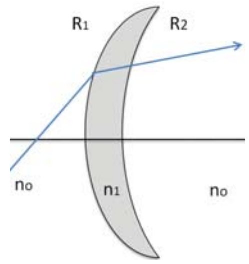

Fig. 3. Thin Lens

4 Applications

In order to illustrate the use and effectiveness of the proposed formalization, we apply it to formally analyze the stability of optical resonators which are very important for various operations of lasers, e.g., alignment sensitivity and beam quality. An optical resonator is a special arrangement of optical components which allows the beam of light to be confined in closed path (Figure 4(a)). The main step to formally analyze a given optical resonator is to construct its formal model using already formalized optical components. We then use our library of formally verified matrices of individual interfaces (plane and spherical) to formally verify the matrix relation of a Z-cavity as shown in Figure 4(b).

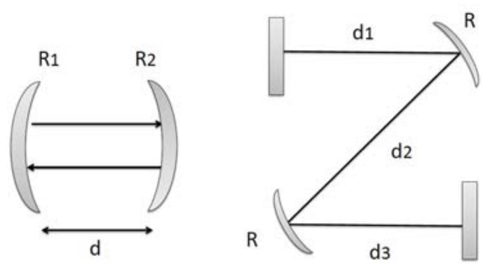

Fig. 4. (a) Simple two mirror resonator (b) The Z-cavity

Theorem 5 (Z-Cavity Matrix).
$\vdash \forall$ R d_1 d_2. R <> 0 $\wedge$ 0 < d_1 $\wedge$ 0 < d_2 $\wedge$ $\Longrightarrow$
system_composition ([(n,0), plane, reflected; (n,d_1),
spherical R_1, reflected; (n,d_2), spherical R_2, reflected; (n,d_1),
plane,reflected], n,0) $\Leftrightarrow$

$$\begin{bmatrix} \frac{R^2-4d_1R-2d_2R+4d_1d_2}{R^2} & \frac{(-2_{d1}-R)(-2d_1d_2+Rd_2+Rd_2+2d_1R)}{R^2} \\ \frac{4(d_2-R)}{R^2} & \frac{R^2-4d_1R-2d_2R+4d_1d_2}{R^2} \end{bmatrix}$$

Now, we sketch the stability analysis of an optical and laser resonator which is an ongoing work. The stability of the resonator means that, after n round trips, beam of light should be confined within the resonator. The last step is to formally derive the stability condition under which the resonator remains stable. The generalized stability condition of a two-mirror optical resonator is given as follows:

$$\forall M. \quad -1 \leq \frac{M_{1,1} + M_{2,2}}{2} \leq 1 \tag{3}$$

where M is the matrix of the optical resonator obtained by the multiplication of the matrices of each optical component which is part of the resonator configuration [10]. The above expression looks very simple, but its formal verification involves rather complex mathematics, e.g., trigonometry and eigenvalue problem solving. A direct application of the above result is in determining the minimum radius of curvature of two mirrors to ensure that a Z-cavity (shown in Figure 4(b)) is stable. It is given as follows:

$$\forall R, d_1, d_2. \quad \left| \frac{R^2 - (4d_1 - 2d_2)R + 4d_1 d_2}{R^2} \right| \leq 1 \tag{4}$$

5 Conclusion

In this extended abstract, we report a novel application of formal methods in analyzing optical and laser systems which is based on geometrical optics. We provided a brief introduction of the current state-of-the-art and highlighted their limitations. Next, we presented an overview of geometrical optics followed by some highlights of our higher-order logic formalization. In order to show the practical effectiveness of our formalization, we presented a sketch of the formal stability analysis of a two-mirror resonator. Our plan is to extend this work in order to obtain an extensive library of verified optical components, along with their ray-transfer matrices, which would allow a practical use of our formalization in industry.

In the current formalization, we use paraxial approximation, i.e., $sin(\theta)$ is treated as θ. In the future, we plan to formally take into account this paraxial approximation using asymptotic notations [1]. We also plan to formally verify Snell's law from the Fermat's principle of least time [17].

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