On the Formal Analysis of Geometrical Optics in HOL

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Abstract. Geometrical optics, in which light is characterized as rays, provides an efficient and scalable formalism for the modeling and analysis of optical and laser systems. The main applications of geometrical optics are in stability analysis of optical resonators, laser mode locking and micro opto-electro-mechanical systems. Traditionally, the analysis of such applications has been carried out by informal techniques like paper-and-pencil proof methods, simulation and computer algebra systems. These traditional techniques cannot provide accurate results and thus cannot be recommended for safety-critical applications, such as corneal surgery, process industry and inertial confinement fusion. On the other hand, higher-order logic theorem proving does not exhibit the above limitations, thus we propose a higher-order logic formalization of geometrical optics. Our formalization is mainly based on existing theories of multivariate analysis in the HOL Light theorem prover. In order to demonstrate the practical effectiveness of our formalization, we present the modeling and stability analysis of some optical resonators in HOL Light.

1 Introduction

Different characterizations of light lead to different fields of optics such as quantum optics, electromagnetic optics, wave optics and geometrical optics. The latter describes light as rays which obey geometrical rules. The theory of geometrical optics can be applied for the modeling and analysis of physical objects with dimensions greater than the wavelength of light. Geometrical optics is based on a set of postulates which are used to derive the rules for the propagation of light through an optical medium. These postulates can be summed up as follows: light travels in the form of rays emitted by a source; an optical medium is characterized by its refractive index; light rays follow Fermat's principle of least time [19].

Optical components, such as thin lenses, thick lenses and prisms are usually centered about an optical axis, around which rays travel at small inclinations (angle with the optical axis). Such rays are called *paraxial rays* and this assumption provides the basis of *paraxial optics* which is the simplest framework of geometrical optics. The paraxial approximation explains how light propagates

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through a series of optical components and provides diffraction-free descriptions of complex optical systems. The change in the position and inclination of a paraxial ray as it travels through an optical system can be efficiently described by the use of matrices [11]. This matrix formalism (called *ray-transfer matrices*) of geometrical optics provides accurate, scalable and systematic analysis of real-world complex optical and laser systems. This fact has led to the widespread usage of ray-transfer matrices in the modeling and analysis of critical physical systems. Typical applications of ray-transfer matrices include analysis of a laser beam propagation through some optical setup [11], the stability analysis of laser or optical resonators [13], laser mode-locking, optical pulse transmission [15] and analysis of micro opto-electro-mechanical systems (MOEMS) [28]. Another promising feature of the matrix formalism of geometrical optics is the prediction of design parameters for physical experiments, e.g., recent dispersion-managed soliton transmission experiment [14] and invention of the first single-cell biological lasers [5].

Traditionally, the analysis of geometrical optics based models has been done using paper-and-pencil proof methods [11,15,14]. However, considering the complexity of present age optical and laser systems, such an analysis is very difficult if not impossible, and thus quite error-prone. Many examples of erroneous paper-and-pencil based proofs are available in the open literature, a recent one can be found in [4] and its identification and correction is reported in [16]. One of the most commonly used computer-based analysis techniques for geometrical optics based models is numerical computation of complex ray-transfer matrices [25,12]. Optical and laser systems involve complex and vector analysis and thus numerical computations cannot provide perfectly accurate results due to the inherent incomplete nature of the underlying numerical algorithms. Another alternative is computer algebra systems [17], which are very efficient for computing mathematical solutions symbolically, but are not 100% reliable due to their inability to deal with side conditions [7]. Another source of inaccuracy in computer algebra systems is the presence of unverified huge symbolic manipulation algorithms in their core, which are quite likely to contain bugs. Thus, these traditional techniques should not be relied upon for the analysis of critical laser and optical systems (e.g., corneal surgery [27]), where inaccuracies in the analysis may even result in the loss of human lives.

In the past few years, higher-order logic theorem proving has been successfully used for the precise analysis of a few continuous physical systems [22,10]. Developing a higher-order logic model for a physical system and analyzing this model formally is a very challenging task since it requires both a good mathematical and physical knowledge. However, it provides an effective way for identifying critical design errors that are often ignored by traditional analysis techniques like simulation and computer algebra systems. We believe that higher-order logic theorem proving [6] offers a promising solution for conducting formal analysis of such critical optical and laser systems. Most of the classical mathematical theories behind geometrical optics, such as Euclidean spaces, multivariate analysis and complex numbers, have been formalized in the HOL Light theorem prover

[8,9]. To the best of our knowledge, the reported formalization of geometrical optics is the first of its kind. Our HOL Light developments of geometrical optics and applications presented in this paper are available for download [20] and thus can be used by other researchers and optics engineers working in industry to conduct the formal analysis of their optical systems. This paper is an extended and improved version of [21].

The rest of the paper is organized as follows: Section 2 describes some fundamentals of geometrical optics, and its commonly used ray-transfer-matrix formalism. Section 3 presents our HOL Light formalization of geometrical optics. In order to demonstrate the practical effectiveness and the use of our work, we present in Section 4 the analysis of two real-world optical resonators: Fabry-Pérot resonator and Z-shaped resonator. Finally, Section 5 concludes the paper and highlights some future directions.

2 Geometrical Optics

When a ray passes through optical components, it undergoes *translation* or *refraction*. When it comes to translation, the ray simply travels in a straight line from one component to the next and we only need to know the thickness of the translation. On the other hand, refraction takes place at the boundary of two regions with different refractive indices and the ray follows the law of refraction, i.e., the angle of refraction relates to the angle of incidence by the relation $n_0 \sin(\phi_0) = n_1 \sin(\phi_1)$, called *Snell's law* [19], where n_0, n_1 are the refractive indices of both regions and ϕ_0, ϕ_1 are the angles of the incident and refracted rays, respectively, with the normal to the surface. In order to model refraction, we thus need the normal to the refracting surface and the refractive indices of both regions.

In order to introduce the matrix formalism of geometrical optics, we consider the propagation of a ray through a spherical interface with radius of curvature R between two mediums of refractive indices n_0 and n_1, as shown in Figure 1. Our goal is to express the relation between the incident and refracted rays. The trajectory of a ray as it passes through various optical components can be specified by two parameters: its distance and angle with the optical axis. Here, the distances with respect to the optical axis of the incident and refracted rays are r_1 and r_0, respectively. Since the thickness of the surface is assumed to be very small, we consider that $r_1 = r_0$. Here, ϕ_0 and ϕ_1 are the angles of the incident and refracted rays with the normal to the spherical surface, respectively. On the other hand, θ_0 and θ_1 are the angles of the incident and refracted rays with the optical axis.

Applying Snell's law at the interface, we have $n_0 \sin(\phi_0) = n_1 \sin(\phi_1)$, which, in the context of the paraxial approximation, reduces to the form $n_0\phi_0 = n_1\phi_1$ since $\sin(\phi) \simeq \phi$ if ϕ is small. We also have $\theta_0 = \phi_0 - \psi$ and $\theta_1 = \phi_1 - \psi$, where ψ is the angle between the surface normal and the optical axis. Since $\sin(\psi) = \frac{r_0}{R}$, then $\psi = \frac{r_0}{R}$ by the paraxial approximation again. We can deduce that:

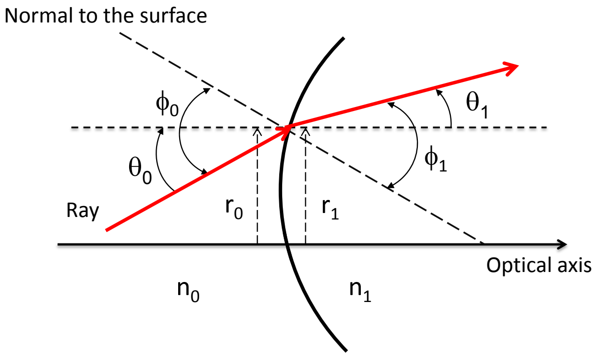


Fig. 1. Spherical interface

$$\theta_1 = \left(\frac{n_0 - n_1}{n_1 R}\right) r_0 + \left(\frac{n_0}{n_1}\right) \theta_0 \tag{1}$$

So, for a spherical surface, we can relate the refracted ray with the incident ray by a matrix relation using equation (1) as follows:

$$\begin{bmatrix} r_1 \\ \theta_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{n_0 - n_1}{n_1 R} & \frac{n_0}{n_1} \end{bmatrix} \begin{bmatrix} r_0 \\ \theta_0 \end{bmatrix}$$

Thus the propagation of a ray through a spherical interface can be described by a 2×2 matrix generally called, in the literature, *ABCD matrix*. It is actually possible to obtain such a 2×2 matrix relating r_1, θ_1 and r_0, θ_0 for many optical components [19].

If we have an optical system consisting of k optical components, then we can trace the input ray R_i through all optical components using multiplication of the matrices of all optical components as follows:

$$R_o = (M_k.M_{k-1}....M_1).R_i \tag{2}$$

where R_o is the output ray and R_i is the input ray.

3 Formalization of Geometrical Optics

In this section, we present a brief overview of our higher-order logic formalization of geometrical optics. The formalization consists of three parts: 1) fundamental concepts of optical systems structures and light ray; 2) frequently used optical components (i.e., thin lens, thick lens and plane parallel plate); 3) optical resonators and their stability.

3.1 Optical System Structure and Ray

The formalization is two-fold: first, we model the geometry and physical parameters of an optical system; second, we model the physical behavior of a ray when it goes through an optical interface. Afterwards, we derive the ray-transfer matrices of the optical components, as explained in Section 2.

An optical system is a sequence of optical components, which consists of free spaces and optical interfaces. We define interfaces by an inductive data type enumerating their different kinds and their corresponding parameters:

Definition 1 (Optical Interface and System)

```
define_type "optical_interface = plane | spherical ℝ"
define_type "interface_kind = transmitted | reflected"
new_type_abbrev ("free_space",`:ℝ × ℝ`)
new_type_abbrev ("optical_component",
            `:free_space × optical_interface × interface_kind`)
new_type_abbrev ("optical_system",
            `:optical_component list × free_space`)
```

An optical component is made of a free space, which is formalized by a pair of real numbers representing the refractive index and width of free space, and an optical interface which is of two kinds: plane or spherical, yielding the corresponding constructors as shown in Figure 2. A spherical interface takes a real number representing its radius of curvature. Finally, an optical system is a list of optical components followed by a free space. When passing through an interface, the ray is either transmitted or reflected. In this formalization, this information is also provided in the type of optical components, as shown by the use of the type `interface_kind`. Note that this datatype can easily be extended to many other optical components if needed.

A value of type `free_space` does represent a real space only if the refractive index is greater than zero. In addition, in order to have a fixed order in the representation of an optical system, we impose that the distance of an optical interface relative to the previous interface is greater or equal to zero. We also need to assert the validity of a value of type `optical_interface` by ensuring that the radius of curvature of spherical interfaces is never equal to zero. This yields the following predicates:

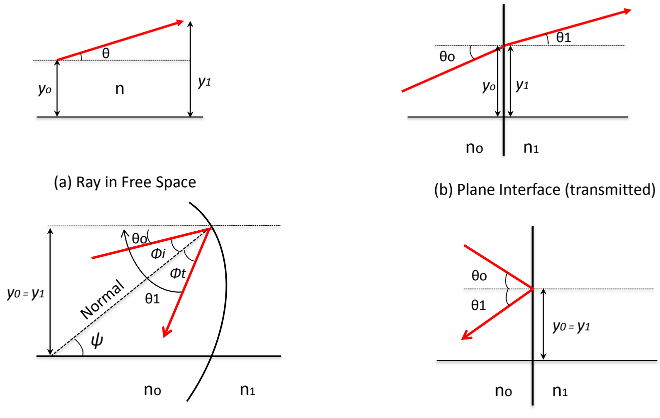
Definition 2 (Valid Free Space and Valid Optical Interface)

```
⊢ is_valid_free_space ((n,d):free_space) ⇔ 0 < n ∧ 0 ≤ d
⊢ (is_valid_interface plane ⇔ T) ∧
  (is_valid_interface (spherical R) ⇔ 0 <> R)
```

Then, by ensuring that this predicate holds for every component of an optical system, we can characterize valid optical systems as follows:

Definition 3 (Valid Optical Component)

```
⊢ ∀fs i ik. is_valid_optical_component ((fs,i,ik):optical_component)
  ⇔ is_valid_free_space fs ∧ is_valid_interface i
```



(a) Ray in Free Space

(b) Plane Interface (transmitted)

(c) Spherical Interface (reflected)

(d) Plane Interface (reflected)

Fig. 2. Behavior of ray at different interfaces

Definition 4 (Valid Optical System)

```
⊢ ∀os fs. is_valid_optical_system ((cs,fs):optical_system) ⇔
  ALL is_valid_optical_component cs ∧ is_valid_free_space fs
```

where `ALL` is a HOL Light library function which checks that a predicate holds for all the elements of a list. We conclude our formalization of an optical system by defining the following helper function to retrieve the refractive index of the first free space in an optical system:

Definition 5 (Refractive Index of First Free Space)

```
⊢ (head_index ([],(n,d)) = n) ∧
  (head_index (CONS ((n,d),i) cs, (nt,dt)) = n)
```

We can now formalize the physical behavior of a ray when it passes through an optical system. We only model the points where it hits an optical interface (instead of modeling all the points constituting the ray). So it is sufficient to just provide the distance of every of these hitting points to the axis and the angle taken by the ray at these points. Consequently, we should have a list of such pairs $(distance, angle)$ for every component of a system. In addition, the same information should be provided for the source of the ray. For the sake of simplicity, we define a type for a pair $(distance, angle)$ as `ray_at_point`. This yields the following definition:

Definition 6 (Ray)

```
new_type_abbrev ("ray_at_point", `:ℝ × ℝ`)
new_type_abbrev ("ray", `:ray_at_point × ray_at_point ×
                        (ray_at_point × ray_at_point) list`)
```

The first `ray_at_point` is the pair $(distance, angle)$ for the source of the ray, the second one is the one after the first free space, and the list of `ray_at_point` pairs represents the same information for the interfaces and free spaces at every hitting point of an optical system. Once again, we specify what is a valid ray by using some predicates. First of all, we define what is the behavior of a ray when it is traveling through a free space. This requires the position and orientation of the ray at the previous and current point of observation, and the free space itself. This is shown in Figure 2(a).

Definition 7 (Behavior of a Ray in Free Space)

$\vdash$ `is_valid_ray_in_free_space` (y_0,θ_0) (y_1,θ_1) `((n,d):free_space)` $\Leftrightarrow$
$y_1 = y_0 + d * \theta_0 \wedge \theta_0 = \theta_1$

Next, we define what is the valid behavior of a ray when hitting a particular interface. This requires the position and orientation of the ray at the previous and current interface, and the refractive index before and after the component. Then the predicate is defined by case analysis on the interface and its type as follows:

Definition 8 (Behavior of a Ray at Given Interface)

$\vdash$ (`is_valid_ray_at_interface` (y_0,θ_0) (y_1,θ_1) n_0 n_1 `plane transmitted`
$\Leftrightarrow y_1 = y_0 \wedge n_0 * \theta_0 = n_1 * \theta_1) \wedge$
(`is_valid_ray_at_interface` (y_0,θ_0) $(y1,\theta_1)$ n_0 n_1 `(spherical R)`
`transmitted` $\Leftrightarrow$ `let` $\phi_i = \theta_0 + \frac{y_1}{R}$ `and` $\phi_t = \theta_1 + \frac{y_1}{R}$ `in`
$y_1 = y_0 \wedge n_0 * \phi_i = n_1 * \phi_t) \wedge$
(`is_valid_ray_at_interface` (y_0,θ_0) (y_1,θ_1) n_0 n_1 `plane reflected`
$\Leftrightarrow y_1 = y_0 \wedge n_0 * \theta_0 = n_0 * \theta_1) \wedge$
(`is_valid_ray_at_interface` (y_0,θ_0) (y_1,θ_1) n_0 n_1 `(spherical R)`
`reflected` $\Leftrightarrow$ `let` $\phi_i = \frac{y1}{R} - \theta_0$ `in` $y_1 = y_0 \wedge \theta_1 = -(\theta_0 + 2 * \phi_i))$

The above definition states some basic geometrical facts about the distance to the axis, and applies Snell's law to the orientation of the ray as shown in Figures 1 and 2. Note that, both to compute the distance and to apply Snell's law, we assumed the paraxial approximation in order to turn $\sin(\theta)$ into θ. Finally, we can recursively apply these predicates to all the components of a system as follows:

Definition 9 (Behavior of a Ray in an Optical System)

$\vdash \forall$ sr_1 sr_2 `h h' fs cs rs i ik` y_0 θ_0 y_1 θ_1 y_2 θ_2 y_3 θ_3 `n d n' d'`.
(`is_valid_ray_in_system` $(sr_1,sr_2,[])$ `(CONS h cs,fs)` $\Leftrightarrow$ `F`) $\wedge$
(`is_valid_ray_in_system` $(sr_1,sr_2,$`CONS h' rs`) `([],fs)` $\Leftrightarrow$`F`)$\wedge$
(`is_valid_ray_in_system` $((y_0,\theta_0),(y_1,\theta_1),[])$ `([],n,d)` $\Leftrightarrow$
`is_valid_ray_in_free_space` (y_0,θ_0) (y_1,θ_1) `(n,d)`) $\wedge$
(`is_valid_ray_in_system` $((y_0,\theta_0),(y_1,\theta_1),$
`CONS` $((y_2,\theta_2),y_3,\theta_3)$ `rs`) `(CONS ((n',d'),i,ik) cs,n,d)` $\Leftrightarrow$
(`is_valid_ray_in_free_space` (y_0,θ_0) (y_1,θ_1) `(n',d')` $\wedge$
`is_valid_ray_at_interface` (y_1,θ_1) (y_2,θ_2) `n'`
`(head_index (cs,n,d)) i ik))` $\wedge$
(`is_valid_ray_in_system` $((y_2,\theta_2),(y_3,\theta_3)$`,rs) (cs,n,d))`

The behavior of a ray going through a series of optical components is thus completely defined. Using this formalization, we verify the ray-transfer matrices as presented in Section 2. In order to facilitate formal reasoning, we define the following matrix relations for free spaces and interfaces.

Definition 10 (Free Space Matrix)

$$\vdash \forall \mathtt{d}.\ \mathtt{free\_space\_matrix\ d} = \begin{bmatrix} 1 & \mathtt{d} \\ 0 & 1 \end{bmatrix}$$

Definition 11 (Interface Matrix)

$\vdash \forall \mathtt{n}_0\ \mathtt{n}_1\ \mathtt{R}.$

$$\mathtt{interface\_matrix}\ \mathtt{n}_0\ \mathtt{n}_1\ \mathtt{plane\ transmitted} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{\mathtt{n}_0}{\mathtt{n}_1} \end{bmatrix} \wedge$$

$$\mathtt{interface\_matrix}\ \mathtt{n}_0\ \mathtt{n}_1\ \mathtt{(spherical\ R)\ transmitted} = \begin{bmatrix} 1 & 0 \\ \frac{\mathtt{n}_0 - \mathtt{n}_1}{\mathtt{n}_0 * \mathtt{R}} & \frac{\mathtt{n}_0}{\mathtt{n}_1} \end{bmatrix} \wedge$$

$$\mathtt{interface\_matrix}\ \mathtt{n}_0\ \mathtt{n}_1\ \mathtt{plane\ reflected} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \wedge$$

$$\mathtt{interface\_matrix}\ \mathtt{n}_0\ \mathtt{n}_1\ \mathtt{(spherical\ R)\ reflected} = \begin{bmatrix} 1 & 0 \\ \frac{-2}{\mathtt{R}} & 1 \end{bmatrix}$$

In the above definition, $\mathtt{n}_0$ and $\mathtt{n}_1$ represent the refractive indices before and after an optical interface. We use the traditional mathematical notation of matrices for the sake of clarity, but, in practice, we use the dedicated functions of HOL Light's vectors library.

Next, we verify the ray-transfer-matrix relation for free spaces:

Theorem 1 (Ray-Transfer-Matrix for Free Space)

$\vdash \forall \mathtt{n\ d}\ \mathtt{y}_0\ \theta_0\ \mathtt{y}_1\ \theta_1.\ \mathtt{is\_valid\_free\_space\ (n,d)} \wedge$
$\mathtt{is\_valid\_ray\_in\_free\_space}\ (\mathtt{y}_0,\theta_0)\ (\mathtt{y}_1,\theta_1)\ \mathtt{(n,d))} \Longrightarrow$

$$\begin{bmatrix} \mathtt{y}_1 \\ \theta_1 \end{bmatrix} = \mathtt{free\_space\_matrix\ d}\ \mathtt{**} \begin{bmatrix} \mathtt{y}_0 \\ \theta_0 \end{bmatrix}$$

where ** represents matrix-vector or matrix-matrix multiplication. The first assumption ensures the validity of free space and the second assumption ensures the valid behavior of ray in free space. The proof of this theorem requires some properties of vectors and matrices along with some arithmetic reasoning. Next, we verify an important theorem describing the general ray-transfer-matrix relation for any interface as follows:

Theorem 2 (Ray-Transfer-Matrix any Interface)

$\vdash \forall \mathtt{n}_0\ \mathtt{n}_1\ \mathtt{y}_0\ \theta_0\ \mathtt{y}_1\ \theta_1\ \mathtt{i\ ik}.\ \mathtt{is\_valid\_interface\ i} \wedge$
$\mathtt{is\_valid\_ray\_at\_interface}\ (\mathtt{y}_0,\theta_0)\ (\mathtt{y}_1,\theta_1)\ \mathtt{n}_0\ \mathtt{n}_1\ \mathtt{i\ ik} \wedge$

$$0 < \mathtt{n}_0 \wedge 0 < \mathtt{n}_1 \Longrightarrow \begin{bmatrix} \mathtt{y}_1 \\ \theta_1 \end{bmatrix} = \mathtt{interface\_matrix}\ \mathtt{n}_0\ \mathtt{n}_1\ \mathtt{i\ ik}\ \mathtt{**} \begin{bmatrix} \mathtt{y}_0 \\ \theta_0 \end{bmatrix}$$

In the above theorem, both assumptions ensure the validity of the interface and behavior of ray at the interface, respectively. This theorem is easily proved by case splitting on `i` and `ik`.

Now, equipped with the above theorem, the next step is to formally verify the ray-transfer-matrix relation for a complete optical system as given in Equation 2. It is important to note that in this equation, individual matrices of optical components are composed in reverse order. We formalize this fact with the following recursive definition:

Definition 12 (System Composition)

```
⊢ system_composition ([],n,d) ⇔ free_space_matrix d ∧
  system_composition (CONS ((nt,dt),i,ik) cs,n,d) ⇔
  (system_composition (cs,n,d) **
  interface_matrix nt (head_index (cs,n,d)) i ik) **
  free_space_matrix dt
```

The general ray-transfer-matrix relation is then given by the following theorem:

Theorem 3 (Ray-Transfer-Matrix for Optical System)

```
⊢ ∀sys ray. is_valid_optical_system sys ∧
  is_valid_ray_in_system ray sys ⟹
  let (y0,θ0),(y1,θ1),rs = ray in
  let yn,θn = last_ray_at_point ray in
```

$$\begin{bmatrix} y_n \\ \theta_n \end{bmatrix} = \texttt{system\_composition sys **} \begin{bmatrix} y_0 \\ \theta_0 \end{bmatrix}$$

Here, the parameters `sys` and `ray` represent the optical system and the ray respectively. The function `last_ray_at_point` returns the last `ray_at_point` of the ray in the system. Both assumptions in the above theorem ensure the validity of the optical system and the good behavior of the ray in the system. The theorem is easily proved by induction on the length of the system and by using previous results and definitions.

This concludes our formalization of optical system structure and ray along with the verification of important properties of optical components and optical systems. The formal verification of the above theorems not only ensures the effectiveness of our formalization but also shows the correctness of our definitions related to optical systems.

3.2 Frequently Used Optical Components

In this section, we present the formal modeling and verification of the ray-transfer matrix relation of thin lens, thick lens and plane parallel plate [19], which are the most widely used components in optical and laser systems.

Generally, lenses are determined by their refractive indices and thickness. A thin lens is represented as the composition of two transmitting spherical interfaces such that any variation of ray parameters (position y and orientation θ) is neglected between both interfaces, as shown in Figure 3 (a). So, at the end, a thin lens is the composition of two spherical interfaces with a null width free space in between. We formalize thin lenses as follows:

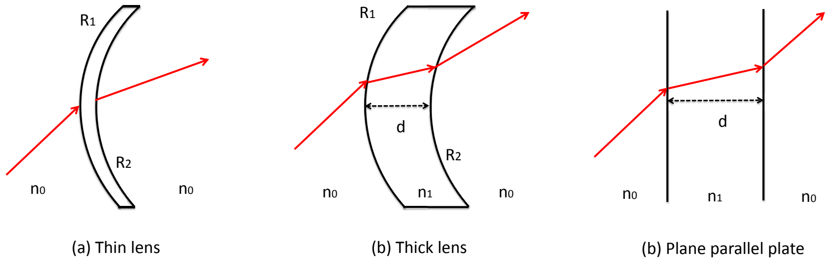


Fig. 3. Frequently used optical components

Definition 13 (Thin Lens)

$\vdash \forall \mathtt{R}_1\ \mathtt{R}_2\ \mathtt{n}_0\ \mathtt{n}_1.\ \mathtt{thin\_lens}\ \mathtt{R}_1\ \mathtt{R}_2\ \mathtt{n}_0\ \mathtt{n}_1 =$
$\mathtt{([(n_0,0),spherical\ R_1,transmitted;\ (n_1,0),spherical\ R_2,transmitted],}$
$\mathtt{(n_0,0))}$

We can then prove that a thin lens is indeed a valid optical system if the corresponding parameters satisfy some constraints:

Theorem 4 (Valid Thin Lens)

$\vdash \forall \mathtt{R}_1\ \mathtt{R}_2\ \mathtt{n}_0\ \mathtt{n}_1.\ \mathtt{R}_1 \neq 0 \wedge \mathtt{R}_2 \neq 0 \wedge 0 < \mathtt{n}_0 \wedge 0 < \mathtt{n}_1 \Longrightarrow$
$\mathtt{is\_valid\_optical\_system\ (thin\_lens\ R_1\ R_2\ n_0\ n_1)}$

Now, in order to simplify the reasoning process, we define the thin lens matrix:

Definition 14 (Thin Lens Matrix)

$$\vdash \forall \mathtt{R}_1\ \mathtt{R}_2\ \mathtt{n}_0\ \mathtt{n}_1.\ \mathtt{thin\_lens\_mat}\ \mathtt{R}_1\ \mathtt{R}_2\ \mathtt{n}_0\ \mathtt{n}_1 = \begin{bmatrix} 1 & 0 \\ \frac{\mathtt{n}_1 - \mathtt{n}_0}{\mathtt{n}_0}(\frac{1}{\mathtt{R}_2} - \frac{1}{\mathtt{R}_1}) & 1 \end{bmatrix}$$

Next, we verify that this matrix is indeed the ray-transfer matrix of the corresponding thin lens:

Theorem 5 (Thin Lens Matrix)

$\vdash \forall \mathtt{R}_1\ \mathtt{R}_2\ \mathtt{n}_0\ \mathtt{n}_1.\ \mathtt{R}_1 \neq 0 \wedge \mathtt{R}_2 \neq 0 \wedge 0 < \mathtt{n}_0 \wedge 0 < \mathtt{n}_1$
$\Longrightarrow \mathtt{system\_composition\ (thin\_lens\ R_1\ R_2\ n_0\ n_1)} =$
$\mathtt{thin\_lens\_mat\ R_1\ R_2\ n_0\ n_1}$

Finally, we can wrap up the behavior of a ray through a thin lens as follows, thanks to Theorem 3:

Theorem 6 (Ray-Transfer-Matrix Model of Thin Lens)

$\vdash \forall \mathtt{R}_1\ \mathtt{R}_2\ \mathtt{n}_0\ \mathtt{n}_1.\ \mathtt{R}_1 \neq 0 \wedge \mathtt{R}_2 \neq 0 \wedge 0 < \mathtt{n}_0 \wedge 0 < \mathtt{n}_1 \Longrightarrow$
$(\forall \mathtt{ray.is\_valid\_ray\_in\_system\ ray\ (thin\_lens\ R_1\ R_2\ n_0\ n_1)}$
$\Longrightarrow \mathtt{(let\ (y_0,}\theta_0\mathtt{),(y1,theta1),rs = ray\ in}$
$\mathtt{(y}_n\mathtt{,}\theta_n\mathtt{) = last\_single\_ray\ ray\ in}$
$\mathtt{vector\ [y}_n\mathtt{;}\theta_n\mathtt{] = thin\_lens\_mat\ R_1\ R_2\ n_0\ n_1\ **\ vector\ [y_0;}\theta_0\mathtt{]))}$

The thick lens is another useful optical component which is used in many real-world optical systems [19]. It is a composition of two spherical interfaces separated by a distance d as shown in Figure 3 (b). We formalize thick lenses as follows:

Definition 15 (Thick Lens)

$\vdash \forall \mathtt{R_1\ R_2\ n_0\ n_1\ d.\ thick\_lens\ R_1\ R_2\ n_0\ n_1\ d =}$
$\mathtt{([(n_0,0),spherical\ R_1,transmitted;\ (n_1,d),spherical\ R_2,transmitted],}$
$\mathtt{(n_0,0))}$

Next, we verify that this lens indeed represents a valid optical system:

Theorem 7 (Valid Thick Lens)

$\vdash \forall \mathtt{R_1\ R_2\ n_0\ n_1.\ R_1} \neq 0 \wedge \mathtt{R_2} \neq 0 \wedge 0 < \mathtt{n_0} \wedge 0 < \mathtt{n_1} \wedge 0 \leq \mathtt{d}$
$\Longrightarrow \mathtt{is\_valid\_optical\_system\ (thick\_lens\ R_1\ R_2\ n_0\ n_1\ d)}$

Again, we define the thick lens matrix:

Definition 16 (Thick Lens Matrix)

$\vdash \forall \mathtt{R_1\ R_2\ n_0\ n_1\ d.\ thin\_lens\_mat\ R_1\ R_2\ n_0\ n_1 =}$

$$\begin{bmatrix} 1 + \frac{d*n_0}{R_1*n_1} - \frac{1}{R1} & \frac{d*n0}{n_0} \\ -\frac{(n_0-n_1)*[d*(n_0\ -\ n_1)\ +\ n_1*(R_1-R_2)]}{n_0*n_1*R_1*R_2} & 1 + d*(\frac{1}{R_2} - \frac{n_1}{n_1*R_2}) \end{bmatrix}$$

We verify that this matrix is indeed the ray-transfer matrix of the thick lens as follows:

Theorem 8 (Thick Lens Matrix)

$\vdash \forall \mathtt{R_1\ R_2\ n_0\ n_1\ d.\ R_1} \neq 0 \wedge \mathtt{R_2} \neq 0 \wedge 0 < \mathtt{n_0} \wedge 0 < \mathtt{n_1} \wedge 0 < \mathtt{n_1} \wedge$
$0 \leq \mathtt{d} \Longrightarrow \mathtt{system\_composition\ (thick\_lens\ R_1\ R_2\ n_0\ n_1) =}$
$\mathtt{thick\_lens\_mat\ R_1\ R_2\ n_0\ n_1}$

We then easily obtain the ray-transfer-matrix relation for thick lenses by using Theorem 3:

Theorem 9 (Ray-Transfer-Matrix Model of Thick Lens)

$\vdash \forall \mathtt{R_1\ R_2\ n_0\ n_1\ d.\ R_1} \neq 0 \wedge \mathtt{R_2} \neq 0 \wedge 0 < \mathtt{n_0} \wedge 0 < \mathtt{n_1} \wedge$
$0 \leq \mathtt{d} \Longrightarrow$
$(\forall\ \mathtt{ray.is\_valid\_ray\_in\_system\ ray\ (thick\_lens\ R_1\ R_2\ n_0\ n_1\ d)}$
$\Longrightarrow \mathtt{(let\ (y_0,\theta_0),(y1,theta1),rs = ray\ in}$
$\mathtt{(y_n,\theta_n) = last\_single\_ray\ ray\ in}$
$\mathtt{vector\ [y_n;\theta_n] = thick\_lens\_mat\ R_1\ R_2\ n_0\ n_1\ d\ **\ vector\ [y_0;\theta_0]))}$

The plane parallel plate is another useful optical component which consists of two plane interfaces separated by some distance d as shown in Figure 3 (c). We formally model plane parallel plates as follows:

Definition 17 (Plane Parallel Plate)

$\vdash \forall \mathtt{n_0\ n_1\ d.\ plane\_parallel\_plate\ n_0\ n_1\ d =}$
$\mathtt{([(n_0,0),plane,transmitted;\ (n_1,d),plane,transmitted],(n_0,0))}$

Next, we verify this system:

Theorem 10 (Valid Plane Parallel Plate)
$\vdash \forall \mathtt{n}_0\ \mathtt{n}_1\ \mathtt{d}.\ 0 < \mathtt{n}_0 \wedge 0 < \mathtt{n}_1 \wedge 0 \leq \mathtt{d} \Longrightarrow$
  `is_valid_optical_system (plane_parallel_plate` $\mathtt{n}_0$ $\mathtt{n}_1$ `d)`

Now, we define the matrix for plane parallel plate:

Definition 18 (Plane Parallel Plate Matrix)
$$\vdash \forall \mathtt{n}_0\ \mathtt{n}_1\ \mathtt{d}.\ \mathtt{plane\_parallel\_mat}\ \mathtt{n}_0\ \mathtt{n}_1\ \mathtt{d} = \begin{bmatrix} 1 & \mathtt{d} * \frac{\mathtt{n}_0}{\mathtt{n}_1} \\ 0 & 1 \end{bmatrix}$$

Next, we verify that this matrix is indeed the ray-transfer matrix of the corresponding plane parallel plate:

Theorem 11 (Plane Parallel Matrix)
$\vdash \forall \mathtt{n}_0\ \mathtt{n}_1\ \mathtt{d}.\ 0 < \mathtt{n}_0 \wedge 0 < \mathtt{n}_1 \wedge 0 \leq \mathtt{d}$
  $\Longrightarrow$ `system_composition (plane_parallel_plate` $\mathtt{n}_0$ $\mathtt{n}_1$ `d) =`
  `plane_parallel_mat` $\mathtt{n}_0$ $\mathtt{n}_1$ `d`

Finally, we can verify the behavior of ray through a plane parallel plate:

Theorem 12 (Ray-Transfer-Matrix Model of Plane Parallel Plate)
$\vdash \forall \mathtt{n}_0\ \mathtt{n}_1\ \mathtt{d}.\ 0 < \mathtt{n}_0 \wedge 0 < \mathtt{n}_1 \wedge 0 \leq \mathtt{d} \Longrightarrow$
  $(\forall$`ray.is_valid_ray_in_system ray (plane_parallel_plate` $\mathtt{n}_0$ $\mathtt{n}_1$ `d)`
    $\Longrightarrow$ `(let (`$\mathtt{y}_0$`,`θ_0`),(y1,theta1),rs = ray in`
      `(`$\mathtt{y}_n$`,`θ_n`) = last_single_ray ray in`
  `vector [`$\mathtt{y}_n$`;`θ_n`] = plane_parallel_mat` $\mathtt{n}_0$ $\mathtt{n}_1$ `d ** vector [`$\mathtt{y}_0$`;`θ_0`]))`

This concludes our formalization of some frequently-used components, which demonstrates how we can use our optics fundamentals formalization in order to define basic systems.

3.3 Optical Resonators and Their Stability

An optical resonator usually consists of mirrors or lenses which are configured in such a way that the beam of light confines in a closed path as shown in Figure 4. Optical resonators are fundamental building blocks of optical devices and lasers. Resonators differ by their geometry and components (interfaces and mirrors) used in their design.

Optical resonators are broadly classified as stable or unstable. Stability analysis identifies geometric constraints of the optical components which ensure that light remains inside the resonator (see Figure 5 (a)). Both stable and unstable resonators have diverse applications, e.g., stable resonators are used in the measurement of refractive index of cancer cells [24], whereas unstable resonators are used in the laser oscillators for high energy applications [23].

The stability analysis of optical resonators involves the study of infinite rays, or, equivalently, of an infinite set of finite rays. Indeed, a resonator is a closed structure terminated by two reflected interfaces and a ray reflects back and forth

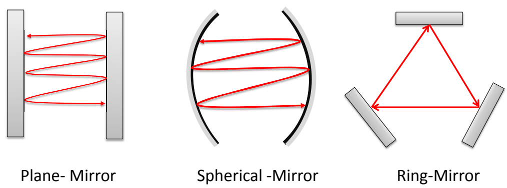


Fig. 4. Optical Resonators

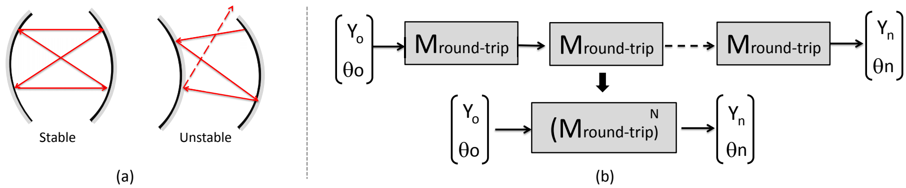


Fig. 5. (a) Types of Optical Resonators (b) Resonator Matrix After N Round-trips

between these interfaces. For example, consider a simple plane-mirror resonator as shown in Figure 4: let m_1 be the first mirror, m_2 the second one, and f the free space in between. Then the stability analysis involves the study of the ray as it goes through f, then reflects on m_2, then travels back through f, then reflects again on m_1, and starts over. So we have to consider the ray going through the "infinite" path $f, m_2, f, m_1, f, m_2, f, m_1, \ldots$, or, using regular expressions notations, $(f, m_2, f, m_1)^*$. Our purpose, regarding stability, is to ensure that this infinite ray remains inside the cavity. This is equivalent to consider that, for every n, the ray going through the path $(f, m_2, f, m_1)^n$ remains inside the cavity. This allows to reduce the study of an infinite path to an infinite set of finite paths.

Our formalization (which is inspired by the way optics engineers model optical systems), presented in Section 3.1, fixes the path of any considered ray. Since we want to consider an infinite set of finite-path rays, we should thus consider an infinite set of optical systems. This has been naturally achieved by optics engineers by "unfolding" the resonator as many times as needed, depending on the considered ray. For instance, consider again the above example of a plane-mirror resonator: if we want to observe a ray going back and forth only once through the cavity, then we should consider the optical system made of f, m_1, f, m_2; however, if we want to study the behavior of rays which make two round-trips through the cavity, then we consider a *new* optical system $f, m_1, f, m_2, f, m_1, f, m_2$; and similarly for more round-trips. This is the standard way optics engineers handle resonators and therefore is the one that we have chosen for our formalization, which we present now.

In our formalization, we want the user to provide only the minimum information so that HOL Light generates automatically the unfolded systems. Therefore, we do not define resonators as just optical systems but define a dedicated type for them: in their most general form, resonators are made of two reflecting interfaces and a list of components in between. We thus define the following type:

Definition 19 (Optical Resonator)

```
define_type "resonator = :interface × optical_component list ×
                          free_space × interface"
```

Note that the additional free space in the type definition is required because the `optical_component` type only contains one free space (the one before the interface, not the one after).

As usual, we introduce a predicate to ensure that a value of type `resonator` indeed models a real resonator:

Definition 20 (Valid Optical Resonator)

```
⊢ ∀i1 cs fs i2. is_valid_resonator ((i1,cs,fs,i2):resonator)⇔
  is_valid_interface i1 ∧ ALL is_valid_optical_component cs ∧
  is_valid_free_space fs ∧ is_valid_interface i1
```

We now present the formalization of the unfolding mentioned above. The first step in this process is to define a function `round_trip` which returns the list of components corresponding to one round-trip in the resonator:

Definition 21 (Round Trip)

```
⊢i1 i1 cs fs. round_trip ((i1,cs,fs,i2):resonator) =
  APPEND cs (CONS (fs,i2,reflected)
  (let cs',fs1 = optical_component_shift cs fs in
  REVERSE (CONS (fs1,i1,reflected) cs')))
```

where `APPEND` is a HOL Light library function which appends two lists, `REVERSE` reverses the order of elements of a list, and `optical_component_shift cs fs` shifts the free spaces of `cs` from right to left, introducing `fs` to the right; the leftmost free space which is "ejected" is also returned by the function. This manipulation is required because unfolding the resonator entails the reversal of the components for the return trip.

We can now define the unfolding of a resonator as follows:

Definition 22 (Unfold Resonator)

```
⊢ unfold_resonator ((i1,cs,fs,i2):resonator) N =
  list_pow (round_trip (i1,cs,fs,i2)) N,(head_index (cs,fs),0)
```

where `list_pow l n` concatenates `n` copies of the list `l`. The argument `N` represents the number of times we want to unfold the resonator. Note that the output type is `optical_system`, therefore all the previous predicates and theorems can be used on an unfolded resonator.

We can now define formally the notion of stability. For an optical resonator to be stable, the distance of the ray from the optical axis and its orientation should remain bounded whatever is the value of N. This is formalized as follows:

Definition 23 (Resonator Stability)

$\vdash \forall$res. is_stable_resonator res $\Leftrightarrow$ ($\forall$r. $\exists$y θ. $\forall$N.
is_valid_ray_in_system r (unfold_resonator res N) $\Longrightarrow$
(let y_n, θ_n = last_single_ray r in abs(y_n) $\leq$ y $\wedge$ abs(θ_n) < θ))

Proving that a resonator satisfies the abstract condition of Definition 23 does not seem trivial at first. However, if the determinant of a resonator matrix M is 1 (which is the case in practice), optics engineers have known for a long time that having $-1 < \frac{M_{11}+M_{22}}{2} < 1$ is sufficient to ensure that the stability condition holds. The obvious advantage of this criterion is that it is immediate to check. This can actually be proved by using Sylvester's Theorem [26], which states that for a matrix M=$\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ such that $\mid M \mid = 1$ and $-1 < \frac{A+D}{2} < 1$, the following holds:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^N = \frac{1}{\sin(\theta)} \begin{bmatrix} A\sin[N\theta] - \sin[(N-1)\theta] & B\sin[N\theta] \\ C\sin[N\theta] & D\sin[N\theta] - \sin[(N-1)\theta] \end{bmatrix}$$

where $\theta = cos^{-1}[\frac{A+D}{2}]$. This theorem allows to prove that stability holds under the considered assumptions: indeed, N only occurs under a sine in the resulting matrix; since the sine itself is comprised between -1 and 1, it follows that the components of the matrix are obviously bounded, hence the stability. We formalize Sylvester's theorem as follows:

Theorem 13 (Sylvester's Theorem)

$\vdash \forall$N A B C D. $\begin{vmatrix} \mathtt{A} & \mathtt{B} \\ \mathtt{C} & \mathtt{D} \end{vmatrix} = 1 \wedge -1 < \frac{\mathtt{A+D}}{2} \wedge \frac{\mathtt{A+D}}{2} < 1 \Longrightarrow$

let $\theta = \mathtt{acs}(\frac{\mathtt{(A+D)}}{2})$ in

$$\begin{bmatrix} \mathtt{A} & \mathtt{B} \\ \mathtt{C} & \mathtt{D} \end{bmatrix}^{\mathtt{N}} = \frac{1}{\mathtt{sin}(\theta)} \begin{bmatrix} \mathtt{A} * \mathtt{sin}[\mathtt{N}\theta] - \mathtt{sin}[(\mathtt{N}-1)\theta] & \mathtt{B} * \mathtt{sin}[\mathtt{N}\theta] \\ \mathtt{C} * \mathtt{sin}[\mathtt{N}\theta] & \mathtt{D} * \mathtt{sin}[\mathtt{N}\theta] - \mathtt{sin}[(\mathtt{N}-1)\theta] \end{bmatrix}$$

We prove Theorem 13 by induction on N and using the fundamental properties of trigonometric functions, matrices and determinants. This allows to derive now the generalized stability theorem for any resonator as follows:

Theorem 14 (Stability Theorem)

$\vdash \forall$res. is_valid_resonator res $\wedge$
($\forall$N. let M = system_composition (unfold_resonator res 1) in
det M = 1 $\wedge$ -1 < $\frac{M_{1,1}+M_{2,2}}{2}$ $\wedge$ $\frac{M_{1,1}+M_{2,2}}{2}$ < 1) $\Longrightarrow$
is_stable_resonator res

where $\mathtt{M}_{\mathtt{i,j}}$ represents the element at column i and row j of the matrix. The formal verification of Theorem 14 requires the definition of stability (Definition 23) and Sylvester's theorem along with the following important lemma:

Lemma 1 (Resonator Matrix)

$\vdash \forall$n res. system_composition (unfold_resonator res N)=
system_composition (unfold_resonator res 1) mat_pow N

where mat_pow is the matrix power function. Note that it is an infix operator. This intuitive lemma formalizes the relation between the unfolding of a resonator and the corresponding ray-transfer matrix.

It is also important to note that our stability theorem is quite general and can be used to verify the stability of almost all kinds of optical resonators.

4 Applications

In this section, we present the stability of two widely used optical resonators: a Fabry Pérot resonator and a Z-shaped resonator.

4.1 Fabry Pérot Resonator

Nowadays, optical systems are becoming more and more popular due to their huge application potential. In order to bring this technology to the market, a lot of research has been done towards the integration of low cost, low power and portable building blocks in optical systems. One of the most important such building blocks is the Fabry Pérot (FP) resonator [19]. Originally, this resonator was used as a high resolution interferometer in astrophysical applications. Recently, the Fabry Pérot resonator has been realized as a microelectromechanical (MEMS) tuned optical filter for applications in reconfigurable Wavelength Division Multiplexing [18]. The other important applications are in the measurement of refractive index of cancer cells [24] and optical bio-sensing devices [2]. As a direct application of the

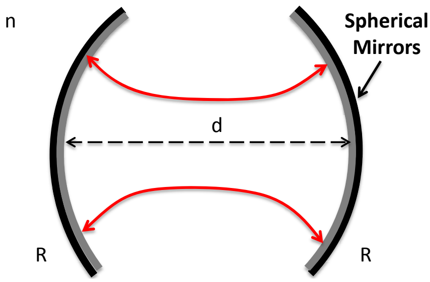


Fig. 6. Fabry Pérot resonator

framework developed in the previous sections, we present the stability analysis of the Fabry Pérot (FP) resonator with spherical mirrors as shown in Figure 6. This architecture is composed of two spherical mirrors with radius of curvature R separated by a distance d and refractive index n. We formally model this resonator as follows:

Definition 24 (FP Resonator)

```
⊢ ∀R d n. (fp_resonator R d n :resonator) =
          (spherical R,[],(n,d),spherical R)
```

where `[]` represents an empty list of components because the given structure has no component between spherical interfaces but only a free space (n,d). Next, we verify that the FP resonator is indeed a valid resonator as follows:

Theorem 15 (Valid FP resonator)
$\vdash \forall$R d n. R $\neq$ 0 $\wedge$ 0 $\leq$ d $\wedge$ 0 < n $\Longrightarrow$
  is_valid_resonator (fp_resonator R d n)

Finally, we formally verify the stability of the FP resonator as follows:.

Theorem 16 (Stability of FP Resonator)
$\vdash \forall$R d n. R $\neq$ 0 $\wedge$ 0 < n $\wedge$ 0 < $\frac{d}{2}$ $\wedge$ $\frac{d}{2}$ < 2 $\Longrightarrow$
  is_stable_resonator (fp_resonator R d n)

The first two assumptions just ensure the validity of the model description. The two following ones provide the intended stability criteria. The formal verification of the above theorem requires Theorem 14 along with some fundamental properties of the matrices and arithmetic reasoning.

4.2 Z-Shaped Resonator

The Z-shaped resonator consists of two plane mirrors and two spherical mirrors as shown in Figure 7. It is widely used in many optical and laser systems including optical bandpass filters and all-optical timing recovery circuits [3]. We formally

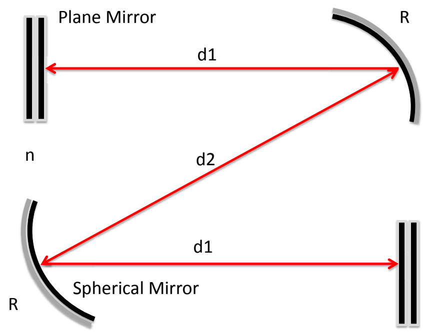


Fig. 7. Z-shaped resonator

model this resonator as follows:

Definition 25 (Z Resonator)
$\vdash \forall$R d_1 d_2 n. (z_resonator R d_1 d_2 n :resonator) = (plane,
 [(n,d_1),spherical R,reflected; (n,d_2),spherical R,reflected],
 (n,d_1),plane)

Here, we have a list of optical components and a free space between two plane mirrors. Again, we check the validity of the Z-shaped resonator as follows:

Theorem 17 (Valid Z-Shaped Resonator)
$\vdash \forall \mathtt{R}\ \mathtt{d}_1\ \mathtt{d}_2\ \mathtt{n}.\ \mathtt{R} \neq 0 \wedge 0 \leq \mathtt{d}_1 \wedge 0 \leq \mathtt{d}_2 \wedge 0 < \mathtt{n} \Longrightarrow$
$\quad \mathtt{is\_valid\_resonator}\ (\mathtt{z\_resonator}\ \mathtt{R}\ \mathtt{d}_1\ \mathtt{d}_2\ \mathtt{n})$

Finally, we formally verify the stability of the FP resonator:

Theorem 18 (Stability of Z-Shaped Resonator)
$\vdash \forall \mathtt{R}\ \mathtt{d}_1\ \mathtt{d}_2\ \mathtt{n}.\ \mathtt{R} \neq 0 \wedge 0 < \mathtt{d}_1 \wedge 0 < \mathtt{n} \wedge \frac{(2*\mathtt{d}_1+\mathtt{d}_2)^2}{2*\mathtt{d}_1} < \mathtt{d}_2 \wedge$
$\quad \frac{2*\mathtt{d}_1*\mathtt{d}_2}{2*\mathtt{d}_1+\mathtt{d}_2} < \mathtt{R} \Longrightarrow \mathtt{is\_stable\_resonator}\ (\mathtt{z\_resonator}\ \mathtt{R}\ \mathtt{d}_1\ \mathtt{d}_2\ \mathtt{n})$

The first three assumptions just ensure the validity of the model description. The two following ones provide the intended stability criteria. The formal verification of the above theorem requires Theorem 14 along with some fundamental properties of the matrices and arithmetic reasoning.

4.3 Discussion

The formal stability analysis of the FP and Z-shaped resonators demonstrates the effectiveness of the proposed theorem proving based approach to reason about geometrical optics. Due to the formal nature of the model and inherent soundness of higher-order logic theorem proving, we have been able to formalize some foundations of geometrical optics along with the verification of some useful theorems about optical components and systems with an unrivaled accuracy. This improved accuracy comes at the cost of the time and effort spent, while formalizing the underlying theory of geometrical optics and resonators. The formalization of geometrical optics, frequently used components and resonators stability took around 1500 lines of HOL Light code and 250 man-hours. But the availability of such a formalized infrastructure significantly reduces the time required for the modeling and stability analysis of FP and Z-shaped resonators as the verification task took just around 100 lines of HOL Light code and a couple of man-hours each. Note that the number of lines has been significantly reduced by the development of some automation tactics, which automatically verifies the validity of a given optical system structure.

5 Conclusion

In this paper, we report a novel application of formal methods in analyzing optical and laser systems which is based on geometrical optics. We provided a brief introduction of the current state-of-the-art and highlighted their limitations. Next, we presented an overview of geometrical optics followed by highlights of our higher-order logic formalization. We also presented the formalization of frequently used optical components like thin lens, thick lens and plane parallel plate. In order to show the practical effectiveness of our formalization, we presented the stability analysis of two widely used optical resonators (i.e., Fabry Pérot resonator and Z-shaped resonator).

Our plan is to extend this work in order to obtain an extensive library of verified optical components, along with their ray-transfer matrices, which would allow a practical use of our formalization in industry. In addition, we plan to formally take into account the paraxial approximation using asymptotic notations [1]. We also intend to improve the traditional stability analysis method by handling infinite paths of rays (as described in Section 3.3) by working directly with all the possible paths of a ray, and thus avoiding the use of unfolding. In particular, this requires a more general treatment of optical interfaces without explicitly mentioning their behavior, i.e., transmitted or reflected. This is a very interesting direction of research since it would even go beyond what optics engineers currently do. Our long term goal is to package our HOL Light formalization in a GUI, so that it can be used by the non-formal methods community in industry for the analysis of practical resonators, and in academia for teaching and research purposes.

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