Towards the Formalization of Fractional Calculus in Higher-Order Logic

Umair Siddique[1]([✉]), Osman Hasan[2], and Sofiène Tahar[1]

[1] Department of Electrical and Computer Engineering,
Concordia University, Montreal, QC, Canada
{muh_sidd,tahar}@ece.concordia.ca
[2] School of Electrical Engineering and Computer Science,
National University of Sciences and Technology, Islamabad, Pakistan
osman.hasan@seecs.nust.edu.pk
http://save.seecs.nust.edu.pk/projects/fc.html

Abstract. Fractional calculus is a generalization of classical theories of integration and differentiation to arbitrary order (i.e., real or complex numbers). In the last two decades, this new mathematical modeling approach has been widely used to analyze a wide class of physical systems in various fields of science and engineering. In this paper, we describe an ongoing project which aims at formalizing the basic theories of fractional calculus in the HOL Light theorem prover. Mainly, we present the motivation and application of such formalization efforts, a roadmap to achieve our goals, current status of the project and future milestones.

Keywords: Fractional calculus · Higher-Order logic · Theorem proving

1 Motivation and Background

Physical and engineering systems are classified as continuous, discrete or hybrid depending upon the nature of underlying system parameters. The rich theories of mathematics provide the necessary tools to study the behaviour of such systems ranging from very small biological organisms to the modern Quantum mechanical phenomenons. Generally, differential equations [39] and difference equations [12] are used to characterize the dynamics of these systems. Consequently, the concept of higher-order differentiation and integration are widely studied in diverse disciplines of science and engineering. For example, it is well understood that the first derivative ($\frac{d}{dt}f(t)$) and second derivative ($\frac{d^2}{dt^2}f(t)$) of a function describe the rate of change and measure of concavity, respectively. However, we rarely think what if the order (n) of higher-order derivative ($\frac{d^n}{dt^n}$) becomes a real, complex or an irrational number? One immediate question arises in our minds is the existence or possibility of such a concept in mathematics. Interestingly, this seemingly new concept dates back to 1695 when L'Hôpital asked Leibniz regarding his notation $\frac{d^n y}{dx^n}$: "*what if n is $\frac{1}{2}$*". In reply, Leibniz [20] prophesied in his letter, "*. . . Thus it follows that $d^{\frac{1}{2}}x$ will be equal to $x\sqrt{dx : x}$.*

This is an apparent paradox from which, one day, useful consequences can be drawn ...". Leibniz's initial work on the problem of defining the derivative of arbitrary order gave birth to a new field of research in mathematics (called *fractional calculus*) and attracted the attention of many physicists, engineers and geometers. Some of the great mathematicians and physicists who touched the field of fractional calculus are Riemann, Liouville, Laurent, Heaviside and Riesz [24].

The concept of fractional calculus has great potential to change the way we model and analyze the systems. It provides good opportunity to scientists and engineers for revisiting the origins. We briefly outline some of the the main applications of fractional calculus in Table 1. The importance of fractional calculus can be realized by the following quote from Miller and Ross [24]. They stated:

> *"... The fractional calculus finds use in many fields of science and engineering, including fluid flow, rheology, diffusive transport akin to diffusion, electrical networks, electromagnetic theory, and probability.... It seems that hardly a field of science or engineering has remained untouched by this topic ... "*

Nowadays engineering systems exhibiting fractional order dynamics are increasingly used in some safety-critical applications such as control systems,

Table 1. Applications of fractional calculus

Field	Applications
Control engineering	- System identification [17]
	- Biomimetic (bionics) control [7]
	- Trajectory control [11]
	- Temperature control [30]
	- Fractional PI^α controller [23]
Signal processing	- Fractional order integrator [19]
	- Fractional order FIR differentiator [37]
	- IIR-type fractional order differentiator [38]
	- Modeling of speech signals [18]
Image processing	- Image restoration and edge detection [29]
	- Satellite image classification [6]
Electromagnetics	- Fractional curl operators [13, 25]
	- Fractional rectangular waveguides [14]
Communication	- Secure chaotic communication [1]
	- Informational network traffic modeling [40]
Biology	- Neuron modeling [5]
	- Biophysical processes [8]
	- Modeling of complex dynamics of tissues [22]

signal processing, electromagnetics and electrical networks (as listed in Table 1).
For example, fractional meta-materials based devices are used to build sensitive
military and defence equipments and electromagnetic stealth technology [21].
Considering these facts, it is quite interesting and important to build a logical
reasoning framework which can be used to formally verify such sophisticated
applications within the sound core of a proof assistant. In fact, proof assistants
have been successfully used to formalize and verify some challenging and para-
doxical mathematical results, e.g., the formal proofs of the Kepler Conjecture
(Flyspeck project) [16] and the Odd Order Theorem [15].

   In this paper, we present details of an ongoing project[1] to develop a formal
reasoning support for fractional calculus in higher-order-logic theorem prover.
This project was originally started at the System Analysis and Verification
(SAVe) lab[2] in 2010. Earlier formalization was done in the HOL4 theorem prover
with the main focus on fractional operators for real-valued functions and the
verification of fractional order electrical components. Later on, the scope of the
project was expanded to formalize fractional calculus involving complex-valued
functions due to its various engineering applications (as listed in Table 1). Cur-
rently, we are using the HOL Light theorem prover due to the availability of
rich multivariate analysis libraries including Harrison's recent formalization of
complex-valued Gamma function[3] as well as the interesting related projects like
Flyspeck [16] and the formalization of optics theories (i.e., ray, wave, electro-
magnetic and quantum) [2].

   The rest of the paper is organized as follows: In Sect. 2, we briefly review some
commonly used notations and definitions of fractional order operators. We pro-
vide an outline of the proposed formalization framework in Sect. 3. Consequently,
the current status of the formalization and future milestones are discussed in
Sect. 4. Finally, we conclude the paper in Sect. 5.

2   Mathematical Framework of Fractional Calculus

There are different notations available for fractional derivatives and integrals. We
use $J_a^v f(x)$ and $D^v f(x)$ for fractional integral and fractional derivative, respec-
tively. In these notations, v is the order of integration or differentiation and a is
the lower limit of integration.

   For every function ($f : \mathbb{C} \to \mathbb{C}$); and for every number $v \in \mathbb{R}$ or $\mathbb{C}$, J_a^v and
D^v should be related to f by the following criteria [10].

1. If $f(x)$ is an analytic function, then $J_a^v f(x)$ and $D^v f(x)$ must also be an
   analytic function of the variable x and of the order v of integration or differ-
   entiation.
2. The operations $J_a^v f(x)$ and $D^v f(x)$ must produce the same result as ordinary
   integration/differentiation when v is a positive integer.

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3. The fractional operators must be linear.

$$J_a^v[\alpha f(x) + \beta g(x)] = \alpha J_a^v f(x) + \beta J_a^v g(x) \tag{1}$$

$$D^v[\alpha f(x) + \beta g(x)] = \alpha D^v f(x) + \beta D^v g(x) \tag{2}$$

4. The operation of order zero must leave the function unchanged.

$$J_a^0 f = f \quad \text{and} \quad D^0 f = f \tag{3}$$

5. The law of exponents must hold for integration and differentiation of arbitrary order under sufficient conditions on function f.

$$J_a^u(J_a^v f) = J_a^{u+v} f \quad \text{and} \quad D^u(D^v f) = D^{u+v} f \tag{4}$$

Fractional integrals and fractional derivatives are also referred to as Differintegrals [27] and there are more than ten well-known definitions for Differintegrals [9]. We describe here two of them, which are most widely used in analyzing real-world problems. These are the Riemann-Liouville and Grünwald-Letnikov definitions, which are also equivalent for a wide class of functions [31].

Riemann-Liouville (RL) Definition:

$$J_a^v f(x) = \frac{1}{\Gamma(v)} \int_a^x (x - t)^{v-1} f(t) dt \tag{5}$$

where $J_a^v f(x)$ represents fractional integration with order v and lower integration limit a. The parameter $a = 0$ gives the Riemann definition and $a = -\infty$ gives the Liouville definition of fractional integration. Indeed Eq. (5) is the generalization of Cauchy's repeated integration formula to non-integer v [32]. Where Γ (.) in the above definition denotes the Gamma function which is defined using the well-known improper integral as follows:

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt \tag{6}$$

for $Re(z) > 0$.

The fractional differentiation is given as follows:

$$D^v f(x) = (\frac{d}{dx})^m J_a^{m-v} f(x) \tag{7}$$

where m represents the ceiling of v, i.e., $\lceil v \rceil$.

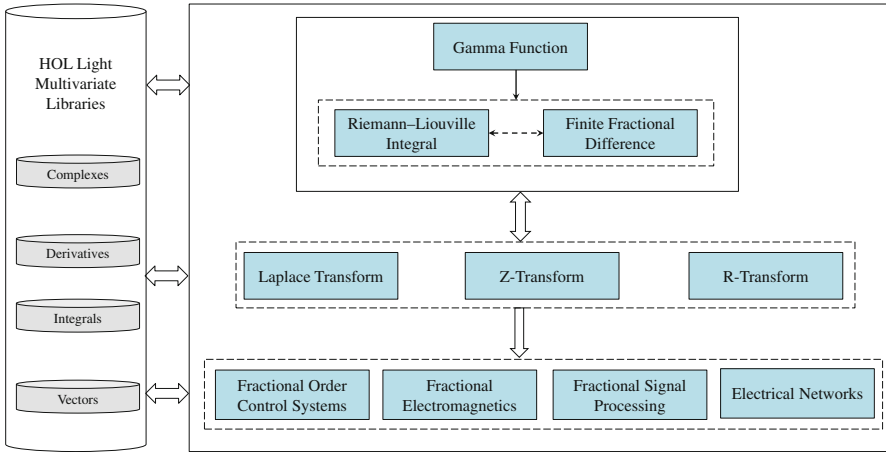**Fig. 1.** Formalization framework for fractional calculus

Grünwald-Letnikov (GL) Definition:

$$_cD_x^v f(x) = \lim_{h \to 0} h^{-v} \sum_{k=0}^{[\frac{x-c}{h}]} (-1)^k \binom{v}{k} f(x - kh) \tag{8}$$

Grünwald-Letnikov definition caters for both fractional differentiation and integration, as positive values of v give fractional differentiation and negative values of v give fractional integration. Here, $\binom{v}{k}$ represents the binomial coefficients, which are described in terms of the Gamma function.

3   Formal Analysis Framework

The proposed framework, given in Fig. 1, outlines the main ideas and roadmap to formalize the basic theory behind fractional calculus. The whole framework can be decomposed mainly into three major parts which are the formalization of the core definitions of fractional order operators, formalization of supporting transformations (i.e., Laplace transform [26], Z-Transform [28] and R-Transform [3]) and engineering applications. The first part heavily relies upon the Gamma function as mentioned in Sect. 2. So the core step is to formalize the Gamma function in higher-order logic (HOL) and verify its important properties. Consequently, any definition of fractional order operators can be formalized in HOL. However, our focus is two main definitions, i.e., Riemann-Liouville (RL) and fractional difference which indeed represent continuous and discrete versions of fractional order operators, respectively. This step also involves the validation of all the properties mentioned in Sect. 2. This requires some important results of

multivariate calculus such as the notion of Lebesgue measurability and Fubini's theorem which provides the reasoning support for iterated and double integrals. Interestingly, both of these requirements are available in the multivariate analysis libraries of HOL Light. The second step is the formalization of important integral transforms which are necessary to analytically solve linear fractional differential and difference equations. We mainly focus on three transforms, namely the Laplace transform, the Z-Transform, and the recently introduced R-Transform [4]. All of these transformations are used to transform complicated fractional differential (or difference) equations to algebraic equations which are easier to manipulate and to deduce interesting properties. Building upon these fundamentals, our ultimate goal is to formally verify a variety of engineering systems including control systems, signal processing, electromagnetics and electrical networks. All formalization steps make use of different multivariate theories of HOL Light, e.g., derivatives, integrals, complex vectors and measure spaces. Finally, the developed libraries of this project will become part of the existing HOL Light libraries.

4   Current Status and Future Milestones

As mentioned earlier, the project initially considered only real-order fractional operators in the HOL4 theorem prover. The main difficulty was the little support to handle improper integrals which was required to formalize the real-valued Gamma function. Therefore, we extended the integration theory of HOL4 by formalizing a variant of improper integrals using sequential limits. This was then used to formalize the Gamma function and verify some of its main properties, such as the pseudo-recurrence relation ($\Gamma(z+1) = z\Gamma(z)$), the functional equation ($\Gamma(1) = 1$) and the factorial generalization ($\Gamma(k+1) = k!$) [34]. We utilized these foundations to formalize Differintegrals, given in Eqs. (5) and (7), which in turn can be used to represent the dynamics of fractional order systems in higher-order logic. We also verified theorems corresponding to some commonly used properties of Differintegrals namely Identity and Linearity. Consequently, we conducted the formal analysis of a fractional order electrical component namely resistoductor, a fractional integrator and a fractional differentiator circuit [33]. Later on, the scope of the project was revised to include complex-valued functions and complex order fractional operators in HOL Light. The main requirement was to formalize the complex-valued Gamma function, Laplace and Z-Transforms. However, the Gamma function[4] was formalized by Harrison in early 2014. In the meantime, we formalized the basic theories of the Laplace transform [36] and the Z-Transform [35]. Currently, we are working on three main topics which include: (1) formal proofs of the uniqueness of Laplace and Z-Transforms which are required to formally verify the inverses of these transforms; (2) vectorial Z-Transform, which extend the simple Z-Transform over complex vectors; and (3) fractional difference equations, which are mainly based on Gamma function,

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[4] https://code.google.com/p/hol-light/source/browse/trunk/Multivariate/gamma.ml.

infinite summations and products over complex functions. Finally, we outline the major tasks to achieve the future milestones as follows:

– Formalization of R-Transform.
– Formalization of Differintegrals for complex-valued functions. This is mainly the generalization of the formalization which was developed in HOL4.
– Formalization of linear fractional differential and difference equations with support to analytical solutions using the transform methods.

During the course of this project, two master and two Ph.D. students have contributed to the formalization. Interestingly, all of them are mainly electrical engineers without prior background of formal methods and higher-order-logic theorem proving. Given the complexity and interdisciplinarity of this research project, it is quite encouraging to see people with an engineering (or physics) background to use proof assistants as a complementary tool. The formalization of the fractional calculus is quite challenging as it requires advanced mathematical concepts of vector integration and Lebesgue measurable functions, etc. So expertise in formal reasoning about these complex mathematical phenomena is required for this formalization, which is quite unique compared to reasoning about software and digital hardware systems. The learning curve of HOL Light varies from student to student. Generally, students start proving basic math equations after a couple of months and the pace of formalization increases over time. Learning HOL Light libraries is not difficult once the basic concepts have been grasped by the user. The formalization of the improper integrals, the Gamma function, the fractional calculus, the Z-Transform and the Laplace transform is approximately 15,000 lines of HOL Light code. One of the major obstacles in the formalization was the identification of suitable mathematical definitions and models. Sometimes, textbook proofs do not follow due to various reasons (corner cases, or the proof steps are too abstract, etc.) and they needed to be re-proved on paper with subtle details. Consequently, we have to modify the definitions and thus change the proofs. But now the current formalization seems quite stable as most of the classical properties have been formally verified for our definitions. We believe that future developments can be built on the foundations that have been formalized as most of the work is for general systems. Finally, another important aspect of this project is the potential to apply developed theories to various applications other than fractional calculus. For example, we demonstrated the use of the Gamma function in probability theory [34], the Z-Transform in signal processing [35], and the Laplace transform in power electronics [36].

5   Conclusion

In this paper, we mainly presented the motivation and ongoing activities of our long term project about the formalization of fractional calculus in the HOL Light theorem prover. The main contribution of this project is a comprehensive framework of formal definitions and theorems about fractional calculus which

can be used to verify modern control, signal processing and electromagnetic systems. Some future directions and recommendations for HOL Light are the improvements in the visualization of proofs, better automation and more accessible tutorials with examples from different engineering/physics topics.

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