On the Formal Analysis of Photonic Signal Processing Systems

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Abstract. Photonic signal processing is an emerging area of research, which provides unique prospects to build high-speed communication systems. Recent advancements in fabrication technology allow on-chip manufacturing of signal processing devices. This fact has lead to the widespread use of photonics in industrial critical applications such as telecommunication, biophotonics and aerospace. One the most challenging aspects in the photonics industry is the accurate modeling and analysis of photonic devices due to the complex nature of light and optical components. In this paper, we propose to use higher-order-logic theorem proving to improve the analysis accuracy by overcoming the known limitations of incompleteness and soundness of existing approaches (e.g., paper-and-pencil based proofs and simulation). In particular, we formalize the notion of transfer function using the signal-flow-graph theory which is the most fundamental step to model photonic circuits. Consequently, we formalize and verify the important properties of the stability and the resonance of photonic systems. In order to demonstrate the effectiveness of the proposed infrastructure, we present the formal analysis of a widely used double-coupler double-ring (DCDR) photonic processor.

Keywords: Photonic signal processing · Signal-flow-graph · Theorem proving · HOL light

1 Introduction

Recent advances in communication technology resulted in the development of sophisticated devices such as multifunction routers and personal digital assistants (PDAs); which brought additional challenges of high-speed, low power and huge bandwidth requirements. However, traditional electronic communication has already reached a point where such issues cannot be addressed. On the other hand, photonics technology offers promising solution to resolve these bottlenecks and provides the better convergence of computation and communication, which is a key to cope with future communication challenges. Although, the complete replacement of existing communication systems is not possible at this point, future communication systems will be based on electronic-photonic convergence as mentioned in the MIT's first Communications Technology Roadmap (CTR)

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[7]. Moreover, some feasibility studies have been conducted to demonstrate the realization of large scale (100,000-node) photonic networks which indicate that photonics has the capabilities to interconnect thousands of computing nodes with an ultimate goal of building Exaflops/second links [19]. The main requirement of designing such systems is to process light waves (counterpart of electronic signals) to achieve the desired functionality such as light amplification, filtering and ultrashort pulse generation. Photonic signal processing (PSP) [5] is an active area of research which offers an efficient framework to process high bandwidth signals with low power consumption. The demand of miniaturized communication devices and recent advances in fabrication technology resulted in the development of very large scale integrated (VLSI) photonic circuits [10]. One of the core steps in photonic systems development life cycle is the physical modeling of fundamental building-blocks such as photonic filters and amplifiers [5]. A significant portion of time is spent finding bugs through the validation of such models in order to minimize the failure risks and monetary loss. In particular, this step is more important in industrial applications, where failures directly lead to safety issues such as in aerospace and biomedical devices. For example, the mission management system of Boeing F/A-18E is linked using a photonic network [25]. In general, there are several aspects of light-wave systems which need to be analyzed; however, the focus of this paper is photonic signal processing which forms the core of modern communication devices.

The first step to analyze the behavior of PSP systems is to obtain the transfer function which relates the input and output signals (light-waves). Consequently, the test for the stability (which ensures that the system output is always finite) and resonance (which ensures the oscillation of light waves at certain frequencies) conditions of the photonic circuit can be identified which are the foremost design criterion. One primary analytical approach is to compute the transfer function by explicitly writing node and loop equations which can further be utilized to analyze some physical aspects (e.g., transfer intensity and dispersion [9]) of photonic systems. Recently, however, the signal-flow graph (SFG) theory (originally proposed by Mason [17]) has been extensively used to compute the transfer function of PSP systems. The main motivation of this choice was inspired by its successful applications in electrical and control systems. Indeed, the problem of finding the transfer function reduces to the computation of the forward paths and loops which further can be plugged into the Mason's gain formula (MSG) [17] (which provides an easy way to find the transfer function). The analysis of complex photonic systems using paper-and-pencil based proofs [5] and computer algorithms [11] is not rigorous and sound and thus cannot be recommended for safety critical applications. We believe that there is a dire need of an accurate framework to build high assurance photonic systems.

The main focus of this paper is to formalize the signal-flow-graph theory along with the Mason's gain formula and strengthen the formal reasoning support in the area of photonic signal processing. Indeed, our current work is at the intersection of two ongoing projects[12], i.e., the formalization of different theories of optics and the formal analysis of signal processing systems. As a first step towards our ultimate goal, we present in this paper the higher-order logic formalization of signal-flow-graph theory and Mason's gain formula for the computation of transfer functions in HOL Light theorem prover [12]. Next, we formalize the notion of stability and resonance along with the formal verification of some important properties such as the finiteness and the cardinality of the set of poles (complex-valued parameters at which the system becomes unstable) and zeros (parameters which determine the resonance condition in the system). In order to show the practical utilization of our work, we formally verify the transfer function of a double-coupler double-ring (DCDR) circuit [5], which is a widely used photonic signal processor. Consequently, we derive the general stability and resonance conditions (for both coherent and incoherent operation [5]), which greatly simplifies the verification for any given DCDR configuration. The rigor of higher-order-logic theorem proving allows us to unveil all the hidden assumptions in the paper-and-pencil based approach reported in [5]. Moreover, we also found some incorrect stability conditions and we formally prove that these conditions lead to an unstable operation of the DCDR circuit. The source code of our formalization is available for download [3] and can be utilized by other researchers and engineers for further developments and the analysis of more practical systems.

The rest of the paper is organized as follows: we highlight the most relevant work about the formal analysis of optical and photonic systems in Section 2. Some fundamentals of signal-flow-graph theory and the Mason's gain formula are reviewed in Section 3. We present the formal analysis framework for the photonic signal processing systems along with highlights of our higher-order logic formalization in Section 4. We describe the analysis of the DCDR photonic processor as an illustrative practical application in Section 5. Finally, Section 6 concludes the paper and provides hints for some future directions.

2 Related Work

In the last decade, formal methods based techniques have been proven to be an effective approach to analyze physical, hybrid and digital engineering systems. Here, we describe the most relevant works for analyzing optical systems using theorem proving. The pioneering work about the formal analysis of optical waveguides has been reported in [13]. However, this work is primarily based on real analysis in HOL4 which is insufficient to capture the dynamics of the real photonic systems which involve complex-valued electric and magnetic fields. In [20], a preliminary infrastructure has been developed in HOL Light to verify some fundamental properties (e.g., ray confinement or stability) of optical systems based on ray optics which can only be used when the size of involved optical components is much larger than the wavelength of light. However, the

[1] http://hvg.ece.concordia.ca/projects/optics/
[2] http://hvg.ece.concordia.ca/projects/signal-processing/

physical meaning of stability considered in [20] and in the current paper are totally different, as the first is related to the ray confinement conditions inside a cavity and later deals with the finite output response. In [22], a preliminary formalization of photonic microresonators has been reported which is only focused towards the transmission and reflection properties of light-waves. This work cannot be used to analyze many signal processing properties of optical systems particularly stability and resonance. A more recent work about quantum formalization of coherent light has been reported in [15], with potential applications in the development of future quantum computers. Other interesting works are the formalization of Laplace transform [23] and Z-transform [21] in the HOL Light. Both of these transformations are less popular in the photonic community due to the additional overhead of transforming back-and-forth from time to frequency domain. On the other hand, most PSP systems can directly be described using the SFG theory where properties of interest (such as stability and resonance) can be analyzed [5]. This is the main motivation of choosing the signal-flow-graph approach to model photonic processing systems in our work.

3 Signal-Flow-Graph Theory and Mason's Gain Formula

A signal-flow graph (SFG) [17] is a special kind of directed graph which is widely used to model engineering systems. Mathematically, it represents a set of linear algebraic equations of the corresponding system. An SFG is a network in which nodes are connected by directed branches. Every node in the network represents a system variable and each branch represents the signal transmission from one node to the other under the assumption that signals flow only in one direction. An example of an SFG is shown in Figure 1 consisting of six nodes. An input (*source node*) and an output (*sink node*) are those which only have outgoing branches and incoming branches, respectively (e.g., node 1 and node 6 in Figure 1). A branch is a directed line from node i to j and the gain of each branch is called the *transmittance* which is represented by t_{ij} as shown in Figure 1. A *path* is a traversal of connected branches from one node to the other and if no node is crossed more than once and it connects the input to the output then the path is called *forward path* otherwise if it leads back to itself without touching any node more than once it is considered as a *feedback path* or a *loop*. The loop containing only one node is called *self loop* and any two loops in the SFG are said to be *touching loops* if they have any common node. The total gain of forward path and a loop can be computed by multiplying the transmittances of each traversed branch.

In the analysis of practical engineering systems, the main task is to characterize the relation among system input and output which is called transfer function. The total transmittance or gain between two given nodes (usually input and output) describes the transfer function of the corresponding system. Mason [17] proposed a computational procedure (also called Mason's gain formula) to obtain the total gain of any arbitrary signal-flow-graph. The formula

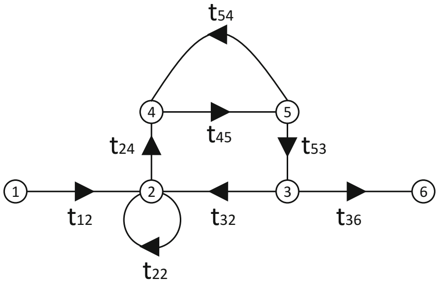

Fig. 1. Signal-Flow-Graph

is described as follows [16]:

$$G = \sum_k \frac{G_k \Delta_k}{\Delta} \tag{1}$$

$$\Delta = 1 - \sum_m P_{m1} + \sum_m P_{m2} - \sum_m P_{m3} + \ldots + (-1)^n \sum \ldots \tag{2}$$

where Δ represents the determinant of the graph, Δ_k represents the value of Δ for the part of graph that is not touching the kth forward path and it is called the cofactor of forward path k, P_{mr} is the gain product of mth possible combination of r non-touching loops. The gain of each forward path is represented by G_k.

4 Proposed Formal Analysis Framework

The proposed framework for the analysis of photonic signal processing systems, given in Figure 2, outlines the necessary steps to encode theoretical fundamentals in higher-order logic. In order to represent a given system in HOL, the first step is the formalization of the signal-flow-graph theory which consists of some new type definitions and the implementation of an algorithm which computes all the elementary circuits (i.e., forward paths and loops). Consequently, this can be used to formalize the Mason's gain formula. The next step is the formalization of the transfer function and its corresponding properties describing different situations such as systems with no forward paths or no touching loops, etc. In order to facilitate the formal modelling of the system properties and reasoning about their satisfaction in the given system model, the last step is to provide the necessary support to express system properties in HOL, i.e., their formal definitions and most frequently used theorems. These system properties are *stability*, which ensures the finite behavior of the system, *resonance*, which provides the basis to derive the suitable parameters at which the photonic circuit can resonate, and *frequency response*, which is necessary to evaluate the frequency dependent system response such as group delay. Finally, we apply the above mentioned steps to develop a library of frequently used photonic signal processing components, such as the double-coupler double-ring [5] or the add-drop filter [26].

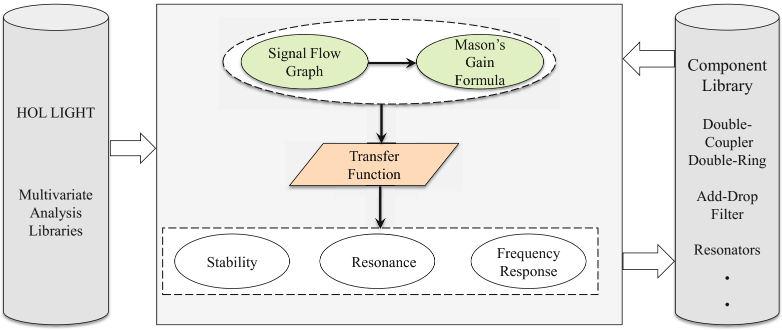

Fig. 2. Proposed Analysis Framework for Photonic Signal Processing Systems

4.1 Formalization of Signal-Flow-Graphs and Mason's Gain

In this section, we only present a brief overview of the formalization developed in our framework (Figure 2). A more detailed description can be found in [4].

We model a single branch as a triplet (a, t_{ab}, b), where a, t_{ab} and b represent the start node, the transmittance and the end node, respectively. Consequently, a path can be modeled as a list of branches and furthermore an SFG can be defined as a composition of a path along with the information about the total number of nodes in the circuit, sink and source nodes at which we want to compute light amplitudes. As mentioned in Section 3, nodes and transmittance represent the system variable and gain, respectively. These parameters are indeed complex valued, i.e., $a, t_{ab}, b \in \mathbb{C}$ in the context of photonic systems. However, the information about the nodes is just used to find properties of signals (light-waves) transmission and they do not appear in the gain and transfer function computation using Mason's gain formula. So, we adopted the same approach as proposed by Mason [17], where nodes of an SFG are represented by natural numbers ($\mathbb{N}$). In order to simplify the reasoning process, we encode the above information by defining three type abbreviations in HOL Light[3], i.e., branch, path and signal-flow-graph as follows:

Definition 1 (Branch, Path and SFG).

```
new_type_abbrev ("branch", `:ℕ × ℂ × ℕ`)
new_type_abbrev ("path", `:(branch)list`)
new_type_abbrev ("sfg", `:path × ℕ × ℕ × ℕ`)
```

where `branch` represents a triplet (a, t_{ab}, b). The second element of `sfg` represents the total number of nodes whereas the third and fourth elements represent the input and output nodes of a signal-flow-graph, respectively.

[3] Note that throughout this paper, we used minimal HOL Light syntax in the presentation of definitions and theorems to improve the readability and for a better understanding without prior experience of HOL Light notations.

Our main task is to find all the forward paths and loops from the source node to the sink node given by the user. We implemented a procedure to extract this information which is mainly inspired from the method proposed in [24]. Briefly, we take an SFG and generate a matrix in which nodes are arranged in the first column and each row represents the branches of the node under consideration. In elementary circuits (loops) extraction, we start the process by the first node of the SFG and go through all possible paths which start from the node under consideration and test for each path whether it is a loop or not. In the next iteration, we go to the next node of the graph and repeat the same process. For forward circuits (forward paths) extraction, we repeat a similar process, but we only consider the paths starting from the source node rather than exploring all the nodes. For the sake of conciseness, we give the following two main definitions of our formalization where more details can be found in [3].

Definition 2 (Elementary Circuits).
⊢ ∀(system : sfg). EC system = if (fst_of_four system = [ ]) then [ ]
else all_loops (EC_MAIN system) system

Here, the function EC_MAIN accepts an SFG, (system : path $\times$ $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$) and returns the list of loops in which each loop is represented as a list of nodes only, and all_loops takes the result of EC_MAIN and an SFG (system) and returns the list of loops in the standard format where each branch represents a triplet. Finally, the main function EC returns an empty list if the system has no branches otherwise it gives the list of all loops in the system.

Definition 3 (Forward Circuits).
⊢ ∀(system : sfg). FC system = if (fst_of_four system = [ ]) then [ ]
else forward_paths (FC_MAIN system) system

where the function FC_MAIN accepts an SFG (system) and returns the list of forward paths in which each forward path is considered as a list of nodes. Then the function forward_paths takes the result of FC_MAIN and system and returns the list of forward paths, such that each forward path is a list of branches.

Finally, we utilize above described definitions to formalize the Mason's gain formula given in Equation 1, as follows:

Definition 4 (Mason's gain formula).
⊢ ∀(system : sfg). Mason_Gain system =

$$\frac{\texttt{product\_gain\_det (EC system) (FC system)}}{\texttt{determinant (EC system)}}$$

where the function Mason_Gain accepts an SFG (system, which is a model of the given system in our case) and computes the Mason's gain as given in Equation 1. Note that the function product_gain_det accepts the list of loops (Definition 2) and forward paths (Definition 3) in the system and computes $\sum_{k \in system} G_k \Delta_k$, where G_k and Δ_k represent, respectively, the product of all forward path gains and the determinant of the k^{th} forward path considering the elimination of all loops touching the k^{th} forward path as described in Section 3. The function `determinant` takes the list of loops and gives the determinant of the system as given in Equation (2).

We developed some simplification tactics for the loops and forward paths extraction and Mason's gain computations. For example, `MASON_SIMP_TAC` accepts a list of theorems (or definitions) and automatically proves or simplify the goal (more details can be found in the source code [3,4]). Next, we present the formalization of the transfer functions which is the second part of the proposed framework (Figure 2).

4.2 Formalization of the Transfer Function

In practice, the physical behavior of any photonic signal processing system is described by the transmittance of each path (or a single branch) involved in the signal-flow-graph. We can consider each path as a system component which processes the input light signal to achieve the desired functionality such as amplification, attenuation or delay [5]. The general expression for the photonic transmittance is given as follows:

$$T_i = t_{a_i} G_i z^{m_i} \tag{3}$$

where i corresponds to the i^{th} path, ta_i is the transmission coefficient for each path expressed as the same path t_a, the parameter G_i is the optical intensity gain factor and m_i is the delay factor of the i^{th} path described as the power of complex-valued parameter z. Note that the parameters t_{a_i} and G_i are constants whereas z is a variable quantity in the system. Indeed, the signal-flow-graph of the given photonic system is expressed as function of z and we need to consider this physical aspect in the formalization of the transfer function which describes the overall behavior of the system. It is mentioned in Section 3 that the Mason's gain formula describes the total gain between the input and the output of the system and hence it can be used to describe the transfer function of the photonic system provided the given signal flow graph can be described as a function of a complex parameter (z). We use the Mason's gain formalization and the above description to formalize the transfer function of a photonic system as follows:

Definition 5 (Photonic System Transfer Function).

$\vdash$ ∀`system. transfer_function system = Mason_Gain (`λ`z. system z)`

where the function `transfer_function` accepts a `system` which has type $\mathbb{C} \rightarrow$ `sfg` and returns a complex ($\mathbb{C}$) quantity which represents the transfer function of the photonic system (`system`). Next, we define the following two helper functions which simplify the formalization of the stability and resonance.

$\vdash$ ∀`sys. numerator sys = product_gain_det (EC sys) (FC sys)`
$\vdash$ ∀`sys. denominator sys = determinant EC sys`

Finally, we verify that any photonic transfer function can be described in terms of the numerator and denominator as follows:

Theorem 1 (Transfer Function).

$$\vdash \forall \texttt{system z. transfer\_function (system z)} = \frac{\texttt{numerator (system z)}}{\texttt{denominator (system z)}}$$

4.3 Formalization of System Properties

To this point, we covered the two components of the proposed framework (Figure 2) which concern the process of formal modeling of the photonic system description provided by the physicists or optical system designers. In order to verify that the given model meets its specification, we need to build the foundations based on which we can formally describe the main system properties (i.e., stability, resonance and frequency response) in HOL. Physically, the stability and resonance are concerned with the identification of all values of z for which the system transfer function becomes infinite and zero, respectively. In the signal processing literature, these values are called *system poles* and *system zeros* which can be computed by the denominator and numerator of the transfer function, respectively. Furthermore, all poles and zeros need to be inside the unit circle which means that their magnitude should be less than 1. The frequency response of the system can be computed by considering the parameter z as a complex exponential $exp(jw)$, where exp, j and w represent the base of logarithm, the imaginary unit $\sqrt{-1}$ and the angular frequency, respectively. We formalize the above mentioned informal description of the system properties in HOL as follows:

Definition 6 (System Poles).

$$\vdash \forall \texttt{system. poles system} = \{\texttt{z} \mid \texttt{z} \neq 0 \wedge \texttt{denominator (system z)} = 0\}$$

$$\vdash \forall \texttt{system. zeros system} = \{\texttt{z} \mid \texttt{z} \neq 0 \wedge \texttt{numerator (system z)} = 0\}$$

where the functions poles and zeros take the system as a parameter and return the set of poles and zeros, respectively. Note that we do not consider the case $\texttt{z} = 0$ because it leads to unconditional stable or resonant system (i.e., 0 is always inside the unit circle). Next, we formalize the notion of stability and resonance as follows:

Definition 7 (System Stability and Resonance).

$$\vdash \forall \texttt{system. is\_stable\_psp system} \Leftrightarrow$$
$$\forall \texttt{p. p} \in \texttt{(poles system)} \implies \parallel \texttt{p} \parallel < 1$$

$$\vdash \forall \texttt{system. is\_resonant\_psp system} \Leftrightarrow$$
$$\forall \texttt{z. z} \in \texttt{(zeros system)} \implies \parallel \texttt{z} \parallel < 1$$

where the predicate is_stable_psp accepts the photonic system (system) and verifies that the magnitude (norm of a complex number, $\parallel \texttt{p}_\texttt{i} \parallel$) of each element

$\mathtt{p_i}$ of the set of poles $\{\mathtt{p_0}, ..., \mathtt{p_n}\}$ is smaller than 1. The function is_resonant_psp is defined in a similar way by considering the zeros of the system.

Next, we verify two important theorems which describe that if the denominator or the numerator of the transfer function is a polynomial of order n, it will always have a finite number of poles or zeros and the cardinality of the set of poles and zeros can only be equal or less than n.

Theorem 2 (Finiteness and Cardinality of Poles).

$$\vdash \forall \mathtt{n\ c\ system}.\ \neg(\forall \mathtt{i.\ i} \in \{0,1,...,\mathtt{n}\} \Rightarrow \mathtt{c\ i} = 0) \wedge$$
$$(\forall \mathtt{z.\ denominator\ (system\ z)} = \textstyle\sum_{\mathtt{i}\in\{0,1,\ldots,\mathtt{n}\}}(\lambda \mathtt{i.\ c\ i} * \mathtt{z^i})) \implies$$
$$\mathtt{FINITE\ (poles\ (system\ z))} \wedge \mathtt{CARD\ (poles\ (system\ z))} \le \mathtt{n}$$

where n represents the order of the complex polynomial function c. The function $\sum_{\mathtt{s}}$ takes two parameters, i.e., s which specifies the set over which the summation occurs and an arbitrary function $\mathtt{f} : (\mathtt{A} \to \mathbb{R}^{\mathtt{N}})$. The functions FINITE and CARD, represent the finiteness and cardinality of a set, respectively. We also prove the same theorem for the set of zeros of a system, where more details can be found in [3]. We formalize the frequency response of a photonic system, group delay and dispersion [8] in terms of the transfer function where more details can be found in [3].

5 Application: Analysis of Photonic Signal Processors

Photonic signal processors process light-waves to achieve different functionalities such as switching, filtering and amplification. In practice, photonic signal processors are of two types (coherent and incoherent) depending upon the nature of light source used in the system. In incoherent photonic processors, the coherence time (i.e., the interval within which the phase of light signal can be predictable) [5] of the light source is much shorter than the unit time delay (or sampling period). On the other hand, coherent processors require the coherence time of the light source to be much longer than the basic time delay to achieve coherent interference of the delayed signals. Both types of photonic processors have wide application domains, e.g., incoherent systems are more stable and mostly used as light amplifiers, whereas coherent integrated optical processors are used in microwave communication systems [5]. The design and analysis of photonic processors mainly involves three steps, i.e., specification of the desired properties of the system, modeling using transfer function and the realization of overall structure (parallel, cascaded, etc.). Given the processor specifications in terms of nature of light sources, transmission powers and optical intensity, the first step is to represent the system as an SFG, the identification of all forward paths and feedback loops and then to compute the system transfer function. Consequently, stability, resonance and frequency response analysis and architectural optimization (possibility of reducing the total number of involved system components) can be performed based on the given specifications. Our proposed framework (Figure 2) allows us to perform these steps (for both coherent and incoherent signal processing) within HOL Light.

The double-coupler double-ring (DCDR) [5] is a widely used processor in the domain of photonics due to its unique features such as compact size, low cost and better compatibility with fiber communication devices. It also has many important physical characteristics due to which it has been used as a photonic filter [5], interferometer [6] and photonic switch [5]. Generally, a DCDR is composed of two main components: (1) Optical directional coupler which are optical devices that transfer the maximum possible optical power from one or more optical devices to another one in a selected direction; and (2) Microring (or cavity) which consists of a fiber ring and confine the light in a very small volume to perform different operations such as light amplification and wavelength filtering.

Using the proposed framework, we formally analyze the DCDR circuit as both coherent and incoherent signal processor. However, we present the analysis of incoherent case while more details about the coherent case can be found in [3]. The schematic diagram of the DCDR circuit is shown in Figure 3 which consists of two directional couplers interconnected with three optical fiber forward and feedback paths. The fiber paths ③-⑥ and ④-⑤ are the forward paths of the circuit while the path ⑦-② is the feedback path of the circuit. The parameters (k_1, k_2), and (T_1,T_2,T_3) represent the power coupling coefficients of the two couplers and the transmission functions of the forward paths, respectively. The photonic transmittance can be expressed as $T_i = t_{a_i} G_i z^{m_i}$ for the i^{th} forward path as described in Section 4.2. The parameters (k_1, k_2) are the deciding factor whether the processor is coherent or incoherent. Typically, for incoherent systems, $k_1 = 1 - k_2$ and for coherent systems $k_1 = \sqrt{1-k}$ and $k_2 = -j\sqrt{k}$, where k is the intensity coupling coefficient [5].

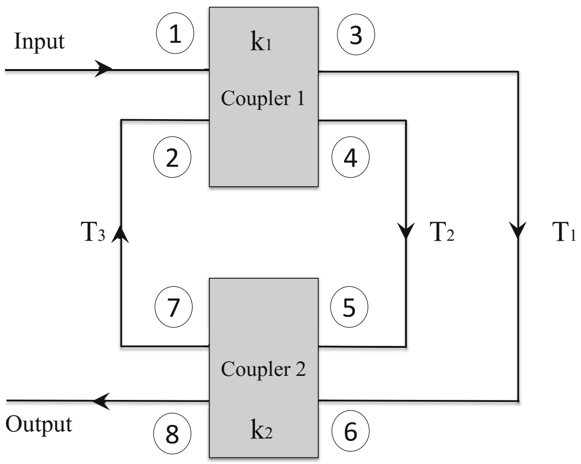

Fig. 3. Double-Coupler Double-Ring Schematic Architecture

The SFG representation of the DCDR circuit is shown in Figure 4 which consists of the same number of nodes as in the block diagram representation in Figure 3. Our main interest is to evaluate the circuit behavior at the output node which is represented by node ⑧, when the signal is applied at the input, i.e., node ①. We keep all above mentioned parameters in general form which further can be used to model different DCDR configurations. We formally define the SFG of the DCDR as follows:

Definition 8 (DCDR Model).

$\vdash \forall \texttt{T}_1\ \texttt{T}_2\ \texttt{T}_3\ \texttt{k}_1\ \texttt{k}_2\ \in \mathbb{C}.$

$\texttt{DCDR\_model}\ \texttt{T}_1\ \texttt{T}_2\ \texttt{T}_3\ \texttt{k}_1\ \texttt{k}_2 = [(1, 1-\texttt{k}_1, 3); (3, \texttt{T}_1, 6); (6, 1-\texttt{k}_2, 8); (1, \texttt{k}_1, 4);$
$(4, \texttt{T}_2, 5); (5, \texttt{k}_2, 8); (6, \texttt{k}_2, 7); (7, \texttt{T}_3, 2); (2, \texttt{k}_1, 3); (2, 1-\texttt{k}_1, 4); (5, 1-\texttt{k}_2, 7)], 8, 1, 8$

where DCDR_model accepts complex-valued transmittances and coupling coefficients, and returns the signal-flow-graph which has a total number of 8 nodes, where 1 and 8 represent the input and output nodes as shown in Figure 3.

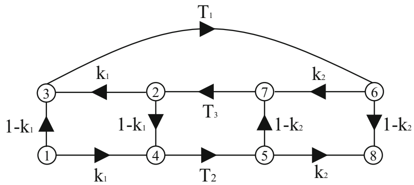

Fig. 4. Signal-Flow-Graph Model of the DCDR

Next, we verify the transfer function of the DCDR circuit as follows:

Theorem 3 (Transfer Function of DCDR).

$\vdash \forall \texttt{T}_1\ \texttt{T}_2\ \texttt{T}_3\ \texttt{k}_1\ \texttt{k}_2\ \in \mathbb{C}.$

$\texttt{transfer\_function}\ (\texttt{DCDR\_model}\ \texttt{T}_1\ \texttt{T}_2\ \texttt{T}_3\ \texttt{k}_1\ \texttt{k}_2) =$

$$\frac{(1-\texttt{k}_1)*(1-\texttt{k}_2)*\texttt{T}_1+\texttt{k}_1*\texttt{k}_2*\texttt{T}_2-(1-2*\texttt{k}_1)*(1-2*\texttt{k}_2)*\texttt{T}_1*\texttt{T}_2*\texttt{T}_3}{1-\texttt{k}_1*\texttt{k}_2*\texttt{T}_1*\texttt{T}_3-(1-\texttt{k}_1)*(1-\texttt{k}_2)*\texttt{T}_2*\texttt{T}_3}$$

The proof of this theorem is mainly based on the extraction of forward paths and loops in the circuit and then using Mason's gain formula. In fact, we developed some simplification tactics [4] which can find elementary and forward circuits to automate the parts of the proof in HOL Light. The transfer function verified in Theorem 3 can be used to analyze four different configurations of DCDR as given in Table 1. One of the most widely used case is when every path has unity delay. Such DCDR circuits are usually used as data processing elements in the photonic communication. The second case of Table 1 describes the conditions when one of the paths in the circuit (Figure 3) amplifies the light signals.

The DCDR circuit operates in passive mode when there is no light amplification in the circuit. Finally, the last case describes the circuit operation when each path can have different delays.

Table 1. DCDR Configurations (parameters G_i and m_i correspond to Eq. 3)

DCDR Configuration	Parameters
Active DCDR Circuit with Unit Delay	$m1 = m_2 = m_3 = 1$
Optical Amplifier in the Fiber Path	$(m1 = m_2 = m_3 = 1) \wedge (G_i > 1)$
Passive DCDR Circuit	$G1 = G_2 = G_3 = 1$
DCDR with Multiple Delay	m_i can have different combinations

In the case of unit delay, the denominator of transfer function of the DCDR can be represented as a second order polynomial which leads to the useful information that the DCDR can have 2 poles at maximum according to Theorem 2. Next, we present the verification of the stability conditions of the DCDR circuit under unit delay conditions as follows:

Theorem 4 (Stability Conditions for Incoherent DCDR).

$$\begin{aligned}
&\vdash \forall \mathtt{G_1}\ \mathtt{G_2}\ \mathtt{G_3}\ \mathtt{k_1}\ \mathtt{k_2}\ \in \mathbb{C}.\\
&\quad \parallel \sqrt{\mathtt{k_1} * \mathtt{k_2} * \mathtt{G_1} * \mathtt{G_2} + (1 - \mathtt{k_1}) * (1 - \mathtt{k_2}) * \mathtt{G_2} * \mathtt{G_3}} \parallel \leq 1 \ \wedge\\
&\quad (\mathtt{k_1} * \mathtt{k_2} * \mathtt{G_1} * \mathtt{G_2} + (1 - \mathtt{k_1}) * (1 - \mathtt{k_2}) * \mathtt{G_2} * \mathtt{G_3}) \neq 0\\
&\quad \Longrightarrow \mathtt{is_stable_psp}\ \ (\lambda \mathtt{z}.\ \mathtt{DCDR}\ (\mathtt{G_1} * \tfrac{1}{\mathtt{z}})\ (\mathtt{G_2} * \tfrac{1}{\mathtt{z}})\ (\mathtt{G_3} * \tfrac{1}{\mathtt{z}})\ \mathtt{k_1}\ \mathtt{k_2})
\end{aligned}$$

where $\parallel . \parallel$ and $\sqrt{.}$ represent the complex norm and complex square root, respectively. The first assumption ensures that both poles are inside the unit circle, whereas the second assumption is required to prove that the poles are indeed valid. Similarly, we verify the second important result, i.e., the resonance condition for the DCDR circuit as follows:

Theorem 5 (Resonance Conditions for Incoherent DCDR).

$$\begin{aligned}
&\vdash \forall \mathtt{G_1}\ \mathtt{G_2}\ \mathtt{G_3}\ \mathtt{k_1}\ \mathtt{k_2} \in \mathbb{C}.\ \parallel \sqrt{\frac{((1-2*\mathtt{k_1})*(1-2*\mathtt{k_2})*\mathtt{G_1}*\mathtt{G_2}*\mathtt{G_3})}{((1-\mathtt{k_1})*(1-\mathtt{k_2})*\mathtt{G_1}+\mathtt{k_1}*\mathtt{k_2}*\mathtt{G_2})}} \parallel \leq 1 \ \wedge\\
&\quad ((1 - 2 * \mathtt{k_1}) * (1 - 2 * \mathtt{k_2}) * \mathtt{G_1} * \mathtt{G_2} * \mathtt{G_3}) \neq 0 \ \wedge\\
&\quad (1 - \mathtt{k_1}) * (1 - \mathtt{k_2}) * \mathtt{G_1} + \mathtt{k_1} * \mathtt{k_2} * \mathtt{G_2}) \neq 0\\
&\quad \Longrightarrow \mathtt{is_resonant_psp}\ \ (\lambda \mathtt{z}.\ \mathtt{DCDR}\ (\mathtt{G_1} * \tfrac{1}{\mathtt{z}})\ (\mathtt{G_2} * \tfrac{1}{\mathtt{z}})\ (\mathtt{G_3} * \tfrac{1}{\mathtt{z}})\ \mathtt{k_1}\ \mathtt{k_2})
\end{aligned}$$

where all assumptions in this theorem are required to ensure that zeros of the DCDR are valid and inside the unit circle.

Similarly, we verify the stability and the resonance conditions of the other DCDR configurations as described in Table 1. One of the main strengths of theorem proving based approach is to unveil all the assumptions under which a theorem can be verified. For example, the second assumption of Theorem 4, and the last two of Theorem 5 are not mentioned in the paper-and-pencil based approach reported in [5]. However, without these assumptions Theorems 4 and 5 cannot be verified. Moreover, our results are verified for universally quantified parameters and the problem of finding the stability and resonance conditions reduces to just ensuring that the values of the system parameters satisfy both assumptions. In an effort to validate the stability results provided in [5], we discovered that both given values of poles cannot satisfy the stability conditions. We formally proved the instability of the DCDR in case of passive operation (i.e., $G1 = G_2 = G_3 = 1$) with $k1 = k2 = 0.9$ as follows:

$\vdash$ `unstable_psp` (λ`z. DCDR` $\frac{1}{z}$ $\frac{1}{z}$ $\frac{1}{z}$ `0.9 0.9` [0.905539; −0.905539])

where `unstable_psp sys` = ¬(`is_stable_psp sys`) as described in Definition 7. This demonstrates the importance of using higher-order-logic theorem proving to unveil such discrepancies. In fact, incorrect stability conditions can lead to the instability of the photonic processor which is hazardous in industrial critical systems which are related to both cost and human safety.

This completes our formal analysis of the DCDR which is a practical photonic processor with vast industrial applications in photonic and microwave communication systems. The stability and resonance conditions have been verified under the general parameters of the DCDR circuit (e.g., k_1, k_2) which is not possible in the case of simulation [5], where these properties are verified for the particular values of k_1 and k_2. Note that the signal-flow-graph model of the DCDR processor involves 8 nodes, however, our formalization is general and can be applied for an arbitrary number of nodes. For example, we formally verified the transfer function of a quadruple optical ring resonator based filter which consists of 20 nodes and 14 complex-valued parameters [8]. We also formalized and verified another important photonic processor namely the add-drop filter [26] which is widely used as a filtering element in biosensors and wavelength division multiplexing (WDM). Some remarkable features of our formalized libraries of SFG and corresponding properties are the generic nature and reusability as the formal specification and verification of above mentioned case studies require minimal efforts. Moreover, we have also made efforts to provide effective automation using derived rules and tactics, so that the application to a particular system does not involve the painful manual proofs often required with interactive (higher-order logic) theorem proving. The source code of the add-drop filter and the quadruple optical ring resonator specification along with their analyses in HOL Light is available at [3]. A brief summary of developed tactics can be found in the Appendix I of [4].

We believe that the formal analysis of above mentioned real-world photonic processors provides two main insights: theorem proving systems have reached to the maturity, where complex physical models can be expressed with less efforts than ever before; and formal methods can assist in the verification of futuristic photonic processors in particular and quantum computers in general. However, the utilization of higher-order-logic theorem proving in industrial settings (particularly, physical systems) is always questionable due to the huge amount of time required to formalize the underlying theories. Another, important factor is the gap between the theorem proving and engineering communities which limits its usage in industry. For example, it is hard to find engineers (or physicists) with theorem proving background and vice-versa. On the other hand, the use of formal methods for safety-critical systems is recommended by different industrial standards like IEC 61508 [14] for electrical and electronics systems, or DO178-B [18] for aviation. In the last decade, some major iconic companies (e.g., Intel [2] and IBM [1]) have established research centers to build revolutionary future computing and communication systems based on the recent advancements in silicon photonics. We believe that applying formal methods to certify photonic designs will be an interesting and challenging future research direction for the formal methods community. Our reported work can be considered as a one step towards an ultimate goal of using theorem provers as a complementary tool in the field of photonics which is one of the rapidly growing high-tech industries in the world today.

6 Conclusion

In this paper, we reported a new application of formal methods in the domain of photonic signal processing. We presented a formal analysis framework based on higher-order logic which provides the required expressiveness and soundness to formally model and verify physical aspects of photonics. In particular, we formalized the signal-flow-graph theory along with Mason's gain formula and transfer functions. Consequently, we presented the formalization of the properties of photonic signal processing systems (such as stability, resonance and frequency response). Finally, we described the formal analysis of the stability and resonance conditions of the double-coupler double-ring photonic processor.

Our immediate future work is to explore the formal relation among the signal-flow-graph representation and the Z-transform [21]. A potential utilization of our formalization and developed automation tactics is to build a framework to certify the results produced by informal tools such as MATLAB based SFG analysis program (available at [11]). Other interesting directions are the application of the current work to formally verify control and digital signal processing systems which are usually modeled as signal-flow-graphs.

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