Towards Ray Optics Formalization of Optical Imaging Systems

Umair Siddique and Sofiène Tahar
Department of Electrical and Computer Engineering, Concordia University, Montreal, Canada
Email: {muh_sidd,tahar}@ece.concordia.ca

Abstract

The verification of optical systems is an important issue due to their safety and financial critical nature (e.g., laser surgeries and space telescopes). Theorem proving offers an attractive solution to overcome the accuracy and soundness problems of traditional approaches like paper-and-pencil based proofs and computer simulation. However, existing formalizations of optics theories do not provide the facility to analyze optical imaging systems which describe the behavior of light ray within the system. In this paper, we present the ray optics formalization of cardinal points which are the most fundamental requirement to model imaging properties of optical systems. We also present the verification of cardinal points for a general system consisting of any number of optical components. For illustration purposes, we present the formal analysis of a thick lens.

1. Introduction

Generally, optical systems consist of a combination of reflecting and refracting surfaces (i.e., mirrors or lenses) to achieve different functionalities such as astronomical imaging, light modulation and short pulse generation. The modeling and analysis of such systems is based on different abstractions of light such as geometrical, wave, electromagnetic and quantum optics. Geometrical or ray optics [15] characterizes light as a set of straight lines which linearly traverse through the optical system. Wave [18] and electromagnetic optics [18] describe the scalar and vectorial wave nature of light, respectively. In Quantum optics [7], light is considered as a stream of photons and electric and magnetic fields are modeled as operators. In general, each of these theories has been used to model different aspects of the same or different optical components. For example, phase-conjugate mirror can be modeled using the ray, electromagnetic and quantum optics. The application of each theory is dependent on the type of system properties which needs to be verified. For example, ray optics provide a convenient way to verify the stability of optical resonators and coupling efficiency of optical fibers. On the other hand, ensuring that no energy is lost when light travels through a waveguide and the analysis of active elements requires electromagnetic and quantum optics theories, respectively. In practice, many optical systems are composed of rotationally symmetric components, i.e., light behavior remains same even the component is rotated along the fixed optical axis [15]. One of the primary design choices is to model a given optical system using the ray optics theory which provides useful information about the overall structure of the system. Moreover, it provides a convenient way to analyze some important properties describing the transformation of input ray (object ray) to the output ray (image ray). For example, some of the properties are the optical power of each component, image size and location etc. In the optics literature, these are called the imaging properties of optical systems. Most of the industrial optical system analysis softwares (e.g., Zemax [14]) provide the facility to analyze such properties.

One of the most challenging requirement in the validation of the practical optical system models is the verification of desired properties. Therefore, a significant portion of time is spent finding design bugs in order to build accurate optical systems. Traditionally, the analysis of optical systems has been done using paper-and-pencil proofs [18]. However, considering the complexity of optical and laser systems, this analysis is very difficult, risky and error-prone. Many examples of erroneous paper-and-pencil proofs are available in the literature of optics (e.g., work reported in [6] was latter corrected in [13]). Another approach is to perform a simulation based analysis of optical systems. This is mainly based on numerical algorithms and suffers from numerical precision and soundness problems. The above mentioned inaccuracy problems of traditional analysis techniques are impeding their usage in designing safety-critical optical systems, where minor bugs can lead to disastrous consequences such as the loss of human lives (e.g., surgeries [12]) or financial loss (e.g., the Hubble Telescope [1], for which the total budget was $1.6 billion). In order to build reliable and accurate optical systems, it is indispensable to develop a framework which is both accurate and scalable for handling complex optical and laser systems.

IEEE IRI 2014, August 13-15, 2014, San Francisco, California, USA
978-1-4799-5880-1/14/$31.00 ©2014 IEEE

Formal methods [19] allow accurate and precise analysis and thus overcome the above mentioned limitations of traditional approaches. The main idea behind them is to develop a mathematical model for the given system and analyze this model using computer-based mathematical reasoning, which in turn increases the chances for catching subtle but critical design errors that are often ignored by traditional techniques. In order to formally verify electronic systems, several formal methods based techniques (such as model checking [5] and theorem proving [9]) have been proposed. Due to the involvement of multivariate calculus (complex linear algebra, complex geometry theory) in the design of optical systems, model checking is not suitable to handle such systems. Recently, some preliminary works for analyzing optical systems using theorem proving [9] have been reported in the literature. For instance, in [11], the formal analysis of optical waveguides using real analysis of HOL4 theorem prover is reported. In [4], complex formalization of electromagnetic optics is reported along with the formalization of quantum mechanics with applications in quantum optics. The preliminary formalization of ray optics is reported in [16, 17] with main applications in the analysis of optical and laser resonators. Despite of the vast applications of optical imaging systems, none of the above mentioned work provide the formalization of basic building-blocks such as the notion of cardinal points [18] (i.e., the pair of points on the optical axis) which are sufficient to completely specify the imaging properties of any geometrical optical system.

The main focus of this paper is to bridge the above mentioned gap and strengthen the formal reasoning support in the area of optical imaging systems. In particular, we build on top of our previous work [17] to formalize composed optical systems which are then utilized to formalize the cardinal points of an arbitrary optical system. Note that this work is a part of an ongoing project[1] to develop a formal reasoning support for different fields of optics (e.g., ray, electromagnetic and quantum optics). In this paper, we use the HOL Light theorem prover [3] to formalize the underlying theories of imaging optical systems. The main reasons of using HOL Light are the existence of rich multivariate analysis libraries [10] as well as active projects like Flyspeck [8].

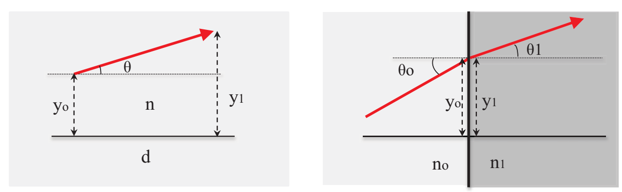
The rest of the paper is organized as follows: Section 2 provides a brief introduction of the ray optics. In Sections 3 and 4, we highlight the formalization of ray optics and composed optical systems. In Section 5, we present the formalization of cardinal points of optical imaging systems. We illustrate the effectiveness of our work by describing the formal modeling and analysis of a thick lens in Section 6. Finally, Section 7 concludes the paper and highlights some future research directions.

[1] http://hvg.ece.concordia.ca/projects/optics/

2. Ray Optics

2.1. Overview

Ray optics describes the propagation of light as rays through different interfaces and mediums. The main governing principle of ray optics is based on some postulates which can be summed up as follows: Light travels in the form of rays emitted by a source; an optical medium is characterized by its refractive index; light rays follow the Fermat's principle of least time [15]. Generally, the main components of optical systems are lenses, mirrors and propagating mediums which is either a free space or some material such as glass. These components are usually centered about an optical axis, around which rays travel at small inclinations (angle with the optical axis). Such rays are called *paraxial rays* and this assumption provides the basis of *paraxial optics* which is the simplest framework of geometrical optics. When a ray passes through optical components, it undergoes *translation* or *refraction*. In translation, the ray simply travels in a straight line from one component to the next and we only need to know the thickness of the translation. On the other hand, refraction takes place at the boundary of two regions with different refractive indices and the ray obeys the law of refraction, called *Paraxial Snell's law* [15]. For example, ray propagation through a free space of width d with refractive index n, and a plane interface (with refractive indices n_1 and n_2, before and after the interface, respectively) is shown in Figure 1



(a) Free Space (b) Plane Interface

Figure 1. Behavior of a Ray at Plane Interface and Free Space

2.2. Modeling Approach

The change in the position and inclination of a paraxial ray as it travels through an optical system can be efficiently described by the use of a matrix algebra. This matrix formalism (called *ray-transfer matrices*) of geometrical optics provides accurate, scalable and systematic analysis of real-world complex optical and laser systems. This is because of the fact that each optical component can be described as (2×2) matrix and many linear algebraic properties can be used in the analysis of optical systems. For example, the general optical system with an input and output ray vector can be described as follows:

$$\begin{bmatrix} y_n \\ \theta_n \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} y_0 \\ \theta_0 \end{bmatrix}$$

Finally, if we have an optical system consisting of N optical components (C_i), then we can trace the input ray R_i through all optical components using the composition of matrices of each optical component as follows:

$$R_o = (C_k.C_{k-1}....C_1).R_i \quad (1)$$

We can write $R_o = M_s R_i$, where $M_s = \prod_{i=k}^{1} C_i$. Here, R_o is the output ray and R_i is the input ray. Similarly, a composed optical system consists of N optical systems inherits the same properties as of a single optical component. This is a very useful modeling notion for the systems which consists of small subsystems due to the already available infrastructure which can be utilized directly with minimal efforts.

2.3. Optical Imaging:

Optical systems capable of being utilized for imaging (can record or transform objects to an image) are called optical imaging systems. Mainly these systems are divided into two main categories, i.e., mirror-systems (also called *catoptrics*, which deal with reflected light rays) and lens-systems (also called *dioptrics*, which deal with refracted light rays). Examples of such systems are optical fibers and telescopes, for the first and second case, respectively. An optical imaging system has many cardinal points which are required to analyze paraxial properties of the optical systems. These points are the *principal points*, the *nodal points* and the *focal points*, which are situated on the optical axis. Figure 2 describes a general optical imaging system with an object point P_0 with a distance x_0 from the optical axis (called the object height). The image is formed by the optical system at point P_1 with a distance x_1 from the optical axis (called the image height). The refractive indices of object space and image space are n and n', respectively. The points F and F' are the foci in the object space and the image space, respectively. The points N and N' are the nodal points in the object and image space. Finally, the points U and U' are the unit or principal points in the object and image space [18].

2.4. Ray Tracing

The propagation of paraxial rays through an optical system is a very useful technique to analyse optical systems. The activity of ray propagation through an optical system is called *ray tracing* [18] and it provides a convenient way for

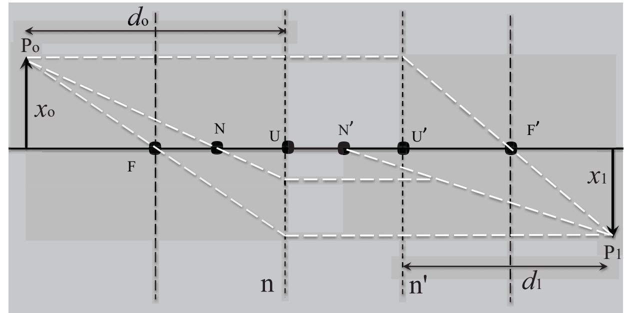


Figure 2. Cardinal Points of an Optical System [18]

the design optimization along with the assessment of imaging quality and properties such as misalignment tolerance and fabrication error analysis of optical components. Ray tracing can be automated and hence it is a part of almost all optical system design tools such as Zemax [14]. There are two types of ray tracing i.e., sequential and non-sequential [18]. In this paper, we only consider sequential ray tracing which is based on three main modeling criterion: First, the nature and parameters of interfaces (plane or spherical) are known (e.g., the radius of curvature in the case of a spherical surface). Second, the position and orientation of each interface is known, i.e., order is fixed. Third, the refractive indices of all components and mediums are known.

On the other hand, in case of non-sequential ray tracing the nature of each interface is not predefined, i.e., at each interface ray can either be transmitted or reflected. However, it is sufficient to consider sequential ray tracing to evaluate the performance of imaging optical systems and hence the main reason of our choice.

3. Formalization of Ray Optics Theory

In this section, we present an overview of our higher-order logic formalization of ray optics. The formalization consists of three parts: A) the formalization of optical system structure; B) the modeling of ray behavior; C) the verification of ray-transfer matrices of optical components. Here, we only provide some highlights of our formalization and more details can be found in [17].

3.1. Modeling of Optical System structure

Ray optics explains the behavior of light when it passes through a free space and interacts with different interfaces like spherical and plane. We can model free space by a pair of real numbers (n, d), which are essentially the refractive index and the total width, as shown in Figure 1 (a). For the sake of simplicity, we consider only two fundamental interfaces, i.e., plane and spherical which are further categorized as either transmitted or reflected. Furthermore, a spherical interface can be described by its radius of curvature (R). We formalize the above description in HOL Light as follows:

Definition 1 (Optical Interface and Free Space)

```
new_type_abbrev ("free_space",`:ℝ × ℝ`)
define_type "optical_interface =
                    plane | spherical ℝ"
define_type "interface_type =
                transmitted | reflected"
```

An optical component is made of a free space (`free_space`) and an optical interface (`optical_interface`) as defined above. Finally, an optical system is a list of optical components followed by a free space. When passing through an interface, the ray is either transmitted or reflected (it is because of the fact that we are only considering sequential ray tracing). In our formalization, this information is also provided in the type definition of optical components, as shown by the use of the type `interface_type` as follows:

Definition 2 (Optical Component and System)

```
new_type_abbrev ("optical_component",
`:free_space × optical_interface ×
interface_type`)
new_type_abbrev ("optical_system",
`:optical_component list
× free_space`)
```

Note that this datatype can easily be extended to many other optical components if needed.

The next step in our formalization is to define some predicates to ensure the validity of free space, optical components and systems. A value of type `free_space` does represent a real space only if the refractive index is greater than zero. In addition, in order to have a fixed order in the representation of an optical system, we impose that the distance of an optical interface relative to the previous interface is greater or equal to zero. We also need to assert the validity of a value of type `optical_interface` by ensuring that the radius of curvature of spherical interfaces is never equal to zero. This yields the following predicates:

Definition 3 (Valid Free Space and Optical Interface)

```
⊢ is_valid_free_space ((n,d):free_space)
   ⇔ 0 < n ∧ 0 ≤ d
⊢ (is_valid_interface plane ⇔ T) ∧
  (is_valid_interface (spherical R)
   ⇔ 0 <> R)
```

Definition 4 (Valid Optical Component)

```
⊢∀fs i ik.  is_valid_optical_component
  ((fs,i,ik):optical_component) ⇔
  is_valid_free_space fs ∧
  is_valid_interface i
```

We can check the validity of an optical system by ensuring that this predicate holds for every component of an optical system.

3.2. Modeling of Ray Behavior

We can now formalize the physical behavior of a ray when it passes through an optical system. We only model the points where it hits an optical interface (instead of modeling all the points constituting the ray). So it is sufficient to just provide the distance of all of these hitting points to the axis and the angle taken by the ray at these points. Consequently, we should have a list of such pairs $(distance, angle)$ for every component of a system. In addition, the same information should be provided for the source of the ray. For the sake of simplicity, we define a type for a pair $(distance, angle)$ as `ray_at_point`. This yields the following definition:

Definition 5 (Ray)

```
new_type_abbrev ("ray_at_point",`:ℝ×ℝ`)
new_type_abbrev ("ray",
  `:ray_at_point × ray_at_point ×
   (ray_at_point × ray_at_point) list`)
```

The first `ray_at_point` is the pair $(distance, angle)$ for the source of the ray, the second one is the one after the first free space, and the list of `ray_at_point` pairs represents the same information for the interfaces and free spaces at every hitting point of an optical system.

Once again, we specify what is a valid ray by using some predicates. First of all, we define what is the behavior of a ray when it is traveling through a free space. In paraxial limit, ray travels in a straight line in free space and thus its distance from the optical axis and angle can be related as $y_1 = y_0 + d * \theta_0$ and $\theta_1 = \theta_0$ (as shown in Figure 1), respectively [15]. In order to model this behavior, we require the position and orientation of the ray at the previous and current point of observation, and the free space itself. We encode above information in HOL Light as follows:

Definition 6 (Behavior of a Ray in Free Space)

```
⊢ is_valid_ray_in_free_space
  (y0,θ0) (y1,θ1) ((n,d):free_space) ⇔
   y1 = y0 + d * θ0 ∧θ0 = θ1
```

where (y_0,θ_0), (y_1,θ_1) and `((n,d):free_space)` represent the ray orientation at previous and current point, and free space, respectively.

Next, we define what is the valid behavior of a ray when hitting a particular interface. This requires the position and orientation of the ray at the previous and current interface, and the refractive indices before and after the component. Then the predicate is defined by case analysis on the interface and its type as follows:

Definition 7 (Behavior of a Ray at Given Interface)

```
⊢ (is_valid_ray_at_interface (y0,θ0)
   (y1,θ1) n0 n1 plane transmitted ⇔
   y1 = y0 ∧ n0 * θ0 = n1 * θ1) ∧
  (is_valid_ray_at_interface
  (y0,θ0) (y1,θ1) n0 n1 (spherical R)
   transmitted ⇔ let φi= θ0 + y1/R and
  φt = θ1 + y1/R in
   y1 = y0 ∧ n0 * φi = n1 * φt) ∧
  (is_valid_ray_at_interface
  (y0,θ0) (y1,θ1) n0 n1 plane reflected
   ⇔ y1 = y0 ∧ n0 * θ0 = n0 * θ1) ∧
  (is_valid_ray_at_interface
  (y0,θ0) (y1,θ1) n0 n1 (spherical R)
   reflected ⇔ let φi = y1/R - θ0 in y1 =
   y0 ∧ θ1 = -(θ0 + 2 * φi))
```

The above definition states some basic geometrical facts about the distance to the axis, and applies paraxial Snell's law [15] to the orientation of the ray. Similarly, we can define the behavior of ray in the entire system by a predicate is_valid_ray_in_system (the definition of this predicate is straightforward, details can be found in [17]).

3.3. Verification of Ray-Transfer Matrices

The main strength of the ray optics is its matrix formulation [18], which provides a convenient way to model all the optical components in the form of a matrix. Indeed, matrix describes a linear relation among input and the output ray. For example, in the case of a free space, the input and output ray parameters are related by two linear equations, i.e., $y_1 = y_0 + d * \theta_0$ and $\theta_1 = \theta_0$, which further can be described in a matrix form as follows:

Theorem 1 (Ray-Transfer-Matrix for Free Space)

```
⊢ ∀n d y0 θ0 y1 θ1.
  is_valid_free_space (n,d) ∧
  is_valid_ray_in_free_space (y0,θ0)
  (y1,θ1) (n,d)) ⟹
```

$$\begin{bmatrix} y_1 \\ \theta_1 \end{bmatrix} = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} ** \begin{bmatrix} y_0 \\ \theta_0 \end{bmatrix}$$

The first assumption ensures the validity of free space and the second assumption ensures the valid behavior of ray in free space. We use the traditional mathematical notation of matrices for the sake of clarity, whereas we define these matrices using the HOL Light Vectors library. We prove the above theorem using the above mentioned definitions and properties of vectors. Similarly, we prove the ray-transfer matrices of plane and spherical interfaces for the case of transmission and reflection (more details can be found in [17].

4. Formalization of Composed Optical Systems

We can trace the input ray R_i through an optical system consisting of N optical components by the composition of ray-transfer matrices of each optical component as described in Equation 1. It is important to note that in this equation, individual matrices of optical components are composed in reverse order. We formalize this fact with the following recursive definition:

Definition 8 (Optical System Model)

```
⊢ optical_system_model ([],n,d) ⇔
  free_space_matrix d ∧
  optical_system_model
  (CONS ((nt,dt),i,ik) cs,n,d) ⇔
  (optical_system_model (cs,n,d) **
  interface_matrix nt
  (head_index (cs,n,d)) i ik) **
  free_space_matrix dt
```

The general ray-transfer-matrix relation is then given by the following theorem:

Theorem 2 (Ray-Transfer-Matrix for Optical System)

```
⊢ ∀sys ray.is_valid_optical_system sys ∧
  is_valid_ray_in_system ray sys ⟹
  let (y0,θ0),(y1,θ1),rs = ray in
  let yn,θn = last_ray_at_point ray in
```

$$\begin{bmatrix} y_n \\ \theta_n \end{bmatrix} = \text{system\_composition sys} ** \begin{bmatrix} y_0 \\ \theta_0 \end{bmatrix}$$

Here, the parameters sys and ray represent the optical system and the ray, respectively. last_ray_at_point returns the last ray_at_point of the ray in the system. Both assumptions in the above theorem ensure the validity of the optical system and the good behavior of the ray in the system. The theorem is easily proved by induction on the length of the system and by using previous results and definitions. Next, we model the notion of composed optical systems as follows:

Definition 9 (Composed Optical System Model)

```
⊢ comp_optical_system
  ([]: optical_system list) ⇔ I ∧
  comp_optical_system
  ((CONS sys) cs) ⇔
  optical_system_model cs **
  optical_system_model sys
```

where I represents the identity matrix.

In order to reason about composed optical systems, we have to define some new definitions similar to the ones presented in the previous section. For example, the validity of a composed optical system means that each of the optical system involved should be valid. Based on this infrastructure, we also verify the ray transfer matrix relation for a composed optical systems similar to the one presented in Theorem 2.

5. Formalization of Cardinal Points

We consider a general optical imaging system as shown in Figure 3. In this context, first and last points of the ray represent the location of object and image. As shown in Figure 3, object (P_0) is located at a distance of d_0 from the optical system and image (P_1) is formed at the distance of d_n. The object and image heights are y_0 and y_n, respectively. The ratio of image height to the object height is called *lateral magnification* which is usually denoted by β. A ray in the object space which intersects the optical axis in the nodal point N at an angle θ intersects the optical axis in the image space in the nodal point N' at the same angle θ'. The ratio of θ and θ' is called *angular magnification*. In our formalization this corresponds to the angle of the first single and and last single ray, respectively. For the sake of geniality, we formalize the general notion of optical system as shown in 3, as follows:

Definition 10 (General Optical System Model)

$\vdash \forall$ sys $\mathtt{d_0}$ $\mathtt{d_n}$ $\mathtt{n_i}$ $\mathtt{n_t}$
gen_optical_system sys $\mathtt{d_0}$ $\mathtt{d_n}$ $\mathtt{n_i}$ $\mathtt{n_t}$ $\Leftrightarrow$
$[([\ ],(\mathtt{n_i},\mathtt{d_0}));\ \ \mathtt{sys};\ \ ([\ ],(\mathtt{n_t},\mathtt{d_n}))]$

Here, the overall system consists of 3 sub-systems, i.e., free space with $(\mathtt{n_i},\mathtt{d_0})$, and general system sys and another free space $(\mathtt{n_t},\mathtt{d_n})$.

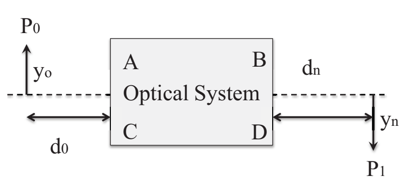


Figure 3. General Optical System [18]

Our next step is to verify the ray-transfer matrix relation of general optical systems by using Theorem 2, as follows:

Theorem 3 (Matrix for General Optical System) $\vdash$
$\forall$sys gray $\mathtt{d_0}$ $\mathtt{d_n}$.
is_valid_optical_system sys $\wedge$
is_valid_ray_in_comp_system gray sys $\wedge$
system_composition sys = $\begin{bmatrix} A & B \\ C & D \end{bmatrix} \Longrightarrow$
let $(y_0,\theta_0),(y_1,\theta_1)$,rs =
fst_single_ray gray in
let y_n,θ_n = last_sng_ray gray in

$$\begin{bmatrix} y_n \\ \theta_n \end{bmatrix} = \begin{bmatrix} A + Cd_n & (Ad_0 + B + Cd_0 d_n + Dd_n) \\ C & Cd_0 + D \end{bmatrix} ** \begin{bmatrix} y_0 \\ \theta_0 \end{bmatrix}$$

Next, we formalize the notion of lateral and angular magnification, as follows:

Definition 11 (Lateral Magnification)

$\vdash \forall$ ray. lateral_magnification ray = $\dfrac{\text{object\_height ray}}{\text{image\_height ray}}$

Definition 12 (Angular Magnification)

$\vdash \forall$ ray. lateral_magnification ray = $\dfrac{\text{object\_angle ray}}{\text{image\_angle ray}}$

where object_height and image_height accept a ray and return the lateral distance of image and object from the optical axis, respectively. Similarly, image_angle and object_angle return the image and object angle, respectively.

The location of all the cardinal points can be found on the optical axis as shown in Figure 2. In case of general optical systems (Figure 3), these can be defined using the distances $\mathtt{d_i}$ and $\mathtt{d_n}$, by developing some constraints.

Principal Points: In order to find principal points, the image has to be formed at the same height as of the object in the object space, i.e., the lateral magnification should be one. This means that all the rays, starting from certain height, will have same height regardless of the incident angle (Mathematically this leads to the fact that second element of 2×2 matrix, representing optical system has to be 0. We package these constraints into the following predicate:

Definition 13 (Principle Points)

$\vdash \forall$ $\mathtt{d_0}$ $\mathtt{d_n}$ $\mathtt{n_i}$ $\mathtt{n_t}$ sys.
are_principle_points $(\mathtt{d_0},\mathtt{d_n})$ $\mathtt{n_i}$ $\mathtt{n_t}$ sys $\Leftrightarrow$
$\forall$ ray. let M = gen_optical_system
sys $\mathtt{d_0}$ $\mathtt{d_n}$ $\mathtt{n_i}$ $\mathtt{n_t}$ and
$\mathtt{y_0}$ = object_height ray and
$\mathtt{y_n}$ = image_height ray in
$\mathtt{y_n} = \mathtt{M_{(1,1)}} * \mathtt{y_0}$ $\wedge$ lateral_magnification ray = 1

where $\mathtt{M_{(i,j)}}$ represents the elements of a square matrix M.

Nodal Points: The second cardinal points of an optical system are the nodal points N (in the object space) and N' (in the image space) as shown in Figure 2. A ray in the object space which intersects the optical axis in the nodal point N at an angle θ intersects the optical axis in the image space at the nodal point N' at the same angle θ', which implies that angular magnification should be 1. We formalize this as follows:

Definition 14 (Nodal Points)

$\vdash \forall$ $\mathtt{d_0\ d_n\ n_i\ n_t\ sys.}$
  are_nodal_points $(\mathtt{d_0}, \mathtt{d_n})$ $\mathtt{n_i\ n_t\ sys} \Leftrightarrow$
  $\forall$ ray. let M = gen_optical_system
      sys $\mathtt{d_0\ d_n\ n_i\ n_t}$ and
  $\mathtt{y_0}$ = object_height ray and
  $\mathtt{y_n}$ = image_height ray in
  $\mathtt{y_0} = 0 \ \wedge \ \mathtt{y_n} = 0 \wedge$ angular_magnification ray $= 1$

Focal Points: The focal points F (in the object space) and F' (in the image space), have two properties: A ray starting from the focus F in the object space is transformed into a ray which is parallel to the optical axis in the image space. Similarly, a ray which is parallel to the optical axis in the object space intersects the focus F' in the image space. We define the following predicate using the above description:

Definition 15 (Focal Points)

$\vdash \forall$ $\mathtt{d_0\ d_n\ n_i\ n_t\ sys.}$
  are_nodal_points $(\mathtt{d_0}, \mathtt{d_n})$ $\mathtt{n_i\ n_t\ sys} \Leftrightarrow$
  $\forall$ ray. let M = gen_optical_system
      sys $\mathtt{d_0\ d_n n_i\ n_t}$ and
  $\mathtt{y_0}$ = object_height ray and
  $\mathtt{y_n}$ = image_height ray and
  θ_0 = object_angle ray and
  θ_n = image_angle ray in
  $\mathtt{y_0} = 0 \ \wedge \ \theta_n = 0 \Rightarrow \mathtt{M}_{(1,2)} * \theta_0 = 0$
  $\mathtt{y_n} = 0 \ \wedge \ \theta_0 = 0 \Rightarrow \mathtt{M}_{(1,1)} * \mathtt{y_0} = 0$

Next, we verify a generic relation describing the cardinal points of the general optical system as follows:

Theorem 4 (Cardinal Points of General System)

$\vdash$ $\mathtt{d_0\ d_n\ n_i\ n_t\ sys.}$
  let M = gen_optical_system
      sys $\mathtt{d_0\ d_n\ n_i\ n_t}$ in
  (are_principle_points
  $((\frac{\mathtt{M}_{(2,2)}}{\mathtt{M}_{(2,1)}} * (\mathtt{M}_{(1,1)} - 1) - \mathtt{B}), (\frac{1-\mathtt{M}_{(1,1)}}{\mathtt{M}_{(2,1)}}))$ $\mathtt{n_i\ n_t\ sys}) \ \wedge$
  (are_nodal_points
  $((\frac{1-\mathtt{M}_{(2,2)}}{\mathtt{M}_{(2,1)}}), (\frac{\mathtt{M}_{(1,1)}}{\mathtt{M}_{(2,1)}} * (\mathtt{M}_{(2,2)} - 1) - \mathtt{B}))$ $\mathtt{n_i\ n_t\ sys}) \ \wedge$
  (are_focal_points
  $((\frac{-\mathtt{M}_{(2,2)}}{\mathtt{M}_{(2,1)}}), (\frac{-\mathtt{M}_{(1,1)}}{\mathtt{M}_{(2,1)}}))$ $\mathtt{n_i\ n_t\ sys})$

This completes the formalization of cardinal points of the optical systems. Theorem 4 is a powerful result as it simplifies the calculation of cardinal points to just finding an equivalent matrix of the given optical system.

6. Application: Verification of the Cardinal Points of Optical Components

In this section, we present the formal verification of the cardinal points of the widely used optical components, i.e., thin lens, thick lens and plane parallel plate [15]. As described in the previous section, the first step is to verify the ray-transfer matrix relation of these components. Consequently, cardinal points can be derived using Theorem 4.

Generally, lenses are characterized by their refractive indices and thickness. A thin lens is represented as the composition of two transmitting spherical interfaces such that any variation of ray parameters (position y and orientation θ) is neglected between both interfaces. At the end, a thin lens is a composition of two spherical interfaces with a free space of null width between them. The thick lens is another useful optical component which is used in many real-world optical systems [15]. It consists of two spherical interfaces separated by a distance d. Finally, the plane parallel plate is another useful optical component which consists of two plane interfaces separated by some distance d. Figure 4 depicts the above description.

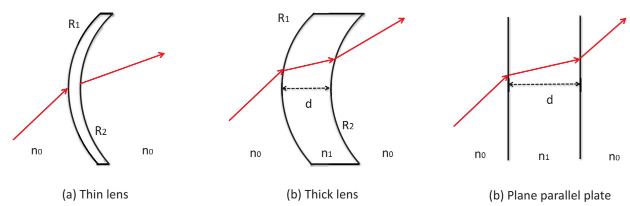


Figure 4. Frequently used optical components

For the sake of conciseness, we only present the formalization of the thick lens as follows: (details about thin lens and plane parallel plate can be found in [17]). We formalize thick lens as follows:

Definition 16 (Thick Lens)

$\vdash \forall \mathtt{R_1\ R_2\ n_0\ n_1\ d.}$
  thick_lens $\mathtt{R_1\ R_2\ n_0\ n_1\ d}$ =
  ([($\mathtt{n_0}$,0),spherical $\mathtt{R_1}$,transmitted;
  ($\mathtt{n_1}$,d),spherical $\mathtt{R_2}$,transmitted],
  ($\mathtt{n_0}$,0))

Next, we verify that thick lens indeed represents a valid optical system:

Theorem 5 (Valid Thick Lens)

$\vdash \forall \mathtt{R_1\ R_2\ n_0\ n_1.}$
  $\mathtt{R_1} \neq 0 \ \wedge \ \mathtt{R_2} \neq 0 \ \wedge \ 0 < \mathtt{n_0} \ \wedge \ 0 < \mathtt{n_1} \ \wedge \ 0 \leq \mathtt{d}$
  $\Longrightarrow$ is_valid_optical_system
  (thick_lens $\mathtt{R_1\ R_2\ n_0\ n_1\ d}$)

We verify the matrix relation for thick lens model (Definition 16) as follows:

Theorem 6 (Thick Lens Matrix)

$\vdash \forall \mathtt{R}_1\ \mathtt{R}_2\ \mathtt{n}_0\ \mathtt{n}_1\ \mathtt{d}.$
$\mathtt{R}_1 \neq 0 \wedge \mathtt{R}_2 \neq 0 \wedge 0 < \mathtt{n}_0 \wedge 0 < \mathtt{n}_1 \wedge 0 < \mathtt{n}_1$
$\wedge\ 0 \leq \mathtt{d} \Longrightarrow \mathtt{system\_composition}$
$\mathtt{(thick\_lens\ R_1\ R_2\ n_0\ n_1)} =$

$$\begin{bmatrix} 1 + \frac{d*n_0}{R_1*n_1} - \frac{1}{R1} & \frac{d*n0}{n_0} \\ -\frac{(n_0-n_1)*[d*(n_0-n_1)+n_1*(R_1-R_2)]}{n_0*n_1*R_1*R_2} & 1 + d*\left(\frac{1}{R_2} - \frac{n_1}{n_1*R_2}\right) \end{bmatrix}$$

The verification of cardinal points of thick lens [18] can be done based on Theorem 4 and Theorem 6 using some rewriting and arithmetic reasoning.

This completes the formal verification of the cardinal points of the optical imaging systems which to the best of our knowledge is first one using theorem proving. Due to the formal nature of the model and the inherent soundness of higher-order logic theorem proving, we have been able to verify generic results such as Theorem 4. This improved accuracy comes at the cost of the time and efforts spent, while formalizing the underlying theory of geometrical optics. Interestingly, the availability of such a formalized infrastructure significantly reduces the time required to analyze the cardinal points of the frequently used optical components. Moreover, we automatized parts of the verification task by introducing new tactics, e.g., `VALID_OPTICAL_SYSTEM_TAC` and `commmon_prove`, which automatically verify the validity of a given optical system and the ray-transfer matrices, respectively. Our HOL Light developments are available for download at [2] and thus can be used by other researchers and optical engineers working in industry to conduct the formal analysis of optical imaging systems.

7. Conclusion

In this paper, we reported a new application of formal methods in the analysis of optical imaging systems based on ray optics. We provided a brief introduction of the current state-of-the-art and highlighted their limitations. Next, we presented an overview of geometrical optics followed by some highlights of our higher-order logic formalization. Consequently, we present the formalization of composed optical systems and cardinal points. In order to show the practical effectiveness of our proposed framework, we presented the verification of the cardinal points of a thick lens.

Our immediate future work is to formalize and verify the correctness and soundness of the ray tracing algorithm [18], which is included in almost all optical systems design tools. Other future directions include the formalization of some more optical components such as parabolic interfaces [18] along with the devolvement of a GUI interface to attract physicists and optical engineers.

References

[1] The Hubble Space Telescope Optical System Failure Report. Technical report, NASA, 1990.

[2] Formal Reasoning about Geometrical Optics, Hardware Verification Group. `http://hvg.ece.concordia.ca/projects/optics/goptics.html`, 2014.

[3] HOL Light. `http://www.cl.cam.ac.uk/~jrh13/hol-light/`, 2014.

[4] S. K. Afshar, U. Siddique, M. Y. Mahmoud, V. Aravantinos, O. Seddiki, O. Hasan, and S. Tahar. Formal Analysis of Optical Systems. *Mathematics in Computer Science*, 8(1):39–70, 2014.

[5] C. Baier and J. P. Katoen. *Principles of Model Checking*. The MIT Press, 2008.

[6] Q. Cheng, T. J. Cui, and C. Zhang. Waves in Planar Waveguide Containing Chiral Nihility Metamaterial. *Optics and Communication*, 274:317–321, 2007.

[7] D. J. Griffiths. *Introduction to Quantum Mechanics*. Pearson Prentice Hall, 2005.

[8] T. C. Hales. Introduction to the Flyspeck Project. In *Mathematics, Algorithms, Proofs*, volume 05021 of *Dagstuhl Seminar Proceedings*, 2005.

[9] J. Harrison. *Handbook of Practical Logic and Automated Reasoning*. Cambridge University Press, 2009.

[10] J. Harrison. The HOL Light Theory of Euclidean Space. *Journal of Automated Reasoning*, 50(2):173–190, 2013.

[11] O. Hasan, S. K. Afshar, and S. Tahar. Formal Analysis of Optical Waveguides in HOL. In *Theorem Proving in Higher Order Logics*, volume 5674 of *LNCS*, pages 228–243. Springer, 2009.

[12] T. Juhasz, G. Djotyan, F. H. Loesel, R. M. Kurtz, C. Horvath, J. F. Bille, and G. Mourou. Applications of Femtosecond Lasers in Corneal Surgery. *Laser Physics*, 10(2):495 – 500, 2011.

[13] A. Naqvi. Comments on Waves in Planar Waveguide Containing Chiral Nihility Metamaterial. *Optics and Communication*, 284:215–216, Elsevier, 2011.

[14] Radiant-Zemax. `http://radiantzemax.com/`, 2014.

[15] B. E. A. Saleh and M. C. Teich. *Fundamentals of Photonics*. John Wiley & Sons, Inc., 1991.

[16] U. Siddique, V. Aravantinos, and S. Tahar. Formal Stability Analysis of Optical Resonators. In *NASA Formal Methods*, volume 7871 of *LNCS*, pages 368–382. Springer, 2013.

[17] U. Siddique, V. Aravantinos, and S. Tahar. On the Formal Analysis of Geometrical Optics in HOL. In *Automated Deduction in Geometry*, volume 7993 of *LNCS*, pages 161–180. Springer, 2013.

[18] F. Träger. *Handbook of Lasers and Optics*. Springer, 2007.

[19] J. Woodcock, P. G. Larsen, J. Bicarregui, and J. Fitzgerald. Formal Methods: Practice and Experience. *ACM Computing Survey*, 41(4):19:1–19:36, 2009.