Formal Stability Analysis of Optical Resonators

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Abstract. An optical resonator usually consists of mirrors or lenses which are configured in such a way that the beam of light is confined in a closed path. Resonators are fundamental components used in many safety-critical optical and laser applications such as laser surgery, aerospace industry and nuclear reactors. Due to the complexity and sensitivity of optical resonators, their verification poses many challenges to optical engineers. Traditionally, the stability analysis of such resonators, which is the most critical design requirement, has been carried out by paper-and-pencil based proof methods and numerical computations. However, these techniques cannot provide accurate results due to the risk of human error and the inherent incompleteness of numerical algorithms. In this paper, we propose to use higher-order logic theorem proving for the stability analysis of optical resonators. Based on the multivariate analysis library of HOL Light, we formalize the notion of light ray and optical system (by defining medium interfaces, mirrors, lenses, etc.). This allows us to derive general theorems about the behaviour of light in such optical systems. In order to illustrate the practical effectiveness of our work, we present the formal analysis of a Fabry-Pérot resonator with fiber rod lens.

1 Introduction

In the last few decades, optical technology has revolutionized our daily life by providing new functionalities and resolving many bottlenecks in conventional electronic systems. The use of optics yields smaller components, high-speed communication and huge information capacity. This provides the basis of miniaturized complex engineering systems including digital cameras, high-speed internet links, telescopes and satellites. Optoelectronic and laser devices based on optical resonators [15] are fundamental building-blocks for new generation, reliable, high-speed and low-power optical systems. Typically, optical resonators are used in lasers [19], optical bio-sensors [1], refractometry [20] and reconfigurable wavelength division multiplexing-passive optical network (WDM-PON) systems [14].

An optical resonator usually consists of mirrors or lenses which are configured in such a way that the beam of light is confined in a closed path as shown in Figure 1. Optical resonators are usually designed to provide high quality-factor and little attenuation [15]. But the most important design requirement is the stability, which states that the beam of light remains within the optical

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resonator even after N round-trips. The stability of a resonator depends on the properties and arrangement of its components, e.g., curvature of mirrors or lenses, and distance between them. For stability analysis, optical resonators are modelled using the principles of geometrical optics [15] which describes light as rays that obey geometrical rules. The theory of geometrical optics can be applied

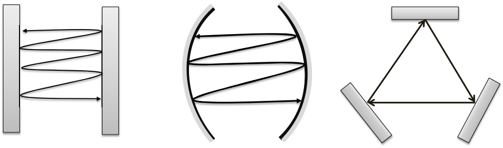

Fig. 1. Optical Resonators

for the modeling and analysis of physical objects with dimensions greater than the wavelength of light. It is based on a set of postulates which are used to derive the rules for the propagation of light through an optical medium. These postulates can be summed up as follows: Light travels in the form of rays emitted by a source; an optical medium is characterized by its refractive index and light rays follow Fermat's principle of least time [15].

Optical components, such as lenses and mirrors are usually centered about an optical axis, around which rays travel at small inclinations (angle with the optical axis). Such rays are called *paraxial rays* and this assumption provides the basis of *paraxial optics* which is the simplest framework of geometrical optics. The change in the position and inclination of a paraxial ray as it travels through an optical system can be described by the use of matrices called *ray-transfer matrices* [19]. This matrix formalism of geometrical optics allows for an accurate, scalable and systematic analysis of real-world complex optical and laser systems.

The widespread use of optical resonators in safety and mission-critical applications, such as astronomy [3] and medicine (e.g., refractive index measurement of cancer cells [20]), poses a real challenge to optical engineers for the modeling and verification of such resonators. Traditionally, the stability analysis of optical resonators has been done using paper-and-pencil based proof methods [10,15,19]. However, considering the complexity of present age optical and laser systems, such an analysis is very difficult if not impossible, and thus quite error-prone. Many examples of erroneous paper-and-pencil based proofs are available in the open literature, a recent one can be found in [2] and its identification and correction is reported in [11]. One of the most commonly used computer-based analysis techniques for stability analysis is numerical computation of complex ray-transfer matrices [13,21,8]. The stability analysis of optical and laser resonators involve complex and vector analysis along with transcendental functions and thus numerical computations cannot provide perfectly accurate results due to the heuristics and approximations of the underlying numerical algorithms. Another alternative is computer algebra systems [12], which are very efficient for computing mathematical solutions symbolically, but are not 100% reliable and sound due to their inability to deal with side conditions [5]. Another source of inaccuracy in computer algebra systems is the presence of unverified huge symbolic manipulation algorithms in their core, which are quite likely to contain bugs. Thus, these traditional techniques should not be relied upon for the analysis of optical resonators which are used in safety-critical applications (e.g., corneal surgery [23]), where inaccuracies in the analysis may even result in the loss of human lives.

In the past few years, higher-order logic theorem proving [4] has been successfully used for the precise analysis of a few continuous physical systems [18]. Developing a higher-order logic model for a physical system and analyzing this model formally is a very challenging task since it requires expertise in both mathematics and physics. However, it provides an effective way for identifying critical design errors that are often ignored by traditional analysis techniques like simulation and computer algebra systems. We believe that higher-order logic theorem proving offers a promising solution for conducting formal analysis of such critical optical resonators. Most of the classical mathematical theories behind geometrical optics, such as Euclidean spaces, multivariate analysis and complex numbers, have been formalized in the HOL Light theorem prover [6,7]. In this paper, we build on our formalization of geometrical optics [16] to provide a practical framework for the stability analysis of optical resonators. In order to illustrate the practical use of our work, we also present the formal analysis of a newly developed Fabry-Pérot resonator with fiber rod lens [10,9]. To the best of our knowledge, the present work is the first one of its kind.

The rest of the paper is organized as follows: Section 2 describes some fundamentals of geometrical optics, and its commonly used ray-transfer-matrix formalism. Section 3 presents the proposed framework for the formal stability analysis of optical resonators. Section 4 presents our HOL Light formalization of geometrical optics. Then, Section 5 describes the formalization of the stability of optical resonators. In order to demonstrate the practical effectiveness and the utilization of the proposed framework, we present the analysis of a real-world optical resonator i.e., Fabry-Pérot resonator with fiber rod lens in Section 6. Finally, Section 7 concludes the paper and highlights some future directions.

2 Geometrical Optics

When a ray passes through optical components, it undergoes *translation* or *refraction*. In translation, the ray simply travels in a straight line from one component to the next and we only need to know the thickness of the translation. On the other hand, refraction takes place at the boundary of two regions with different refractive indices and the ray obeys the law of refraction, i.e., the angle of refraction relates to the angle of incidence by the relation $n_0 \sin(\phi_0) = n_1 \sin(\phi_1)$, called *Snell's law* [15], where n_0, n_1 are the refractive indices of both regions and ϕ_0, ϕ_1 are the angles of the incident and refracted rays, respectively, with the normal to the surface. In order to model refraction, we thus need the normal to the refracting surface and the refractive indices of both regions.

In order to introduce the matrix formalism of geometrical optics, we consider the propagation of a ray through a spherical interface with radius of curvature R between two mediums of refractive indices n_0 and n_1, as shown in Figure 2. Our goal is to express the relationship between the incident and refracted rays. The trajectory of a ray as it passes through various optical components can be specified by two parameters: its distance from the optical axis and its angle with the optical axis. Here, the distances of the incident and refracted rays are r_1 and r_0, respectively, and $r_1 = r_0$ because the thickness of the surface is assumed to be very small. Here, ϕ_0 and ϕ_1 are the angles of the incident and refracted rays with the normal to the spherical surface, respectively. On the other hand, θ_0 and θ_1 are the angles of the incident and refracted rays with the optical axis. Applying Snell's law at the interface, we have $n_0 \sin(\phi_0) = n_1 \sin(\phi_1)$, which, in

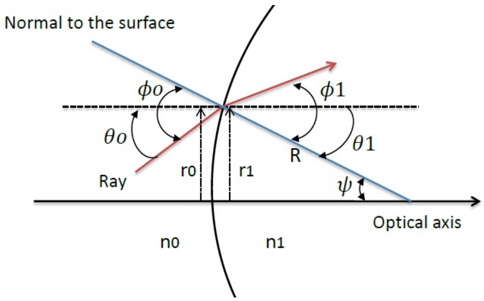

Fig. 2. Spherical Interface

the context of paraxial approximation (i.e., the assumption that light travels at small angles with respect to the normal, which is indeed the case in practice), reduces to the form $n_0\phi_0 = n_1\phi_1$ since $\sin(\phi) \simeq \phi$ if ϕ is small. We also have $\theta_0 = \phi_0 - \psi$ and $\theta_1 = \phi_1 - \psi$, where ψ is the angle between the surface normal and the optical axis. Since $\sin(\psi) = \frac{r_0}{R}$, then $\psi = \frac{r_0}{R}$ by paraxial approximation. We can deduce that:

$$\theta_1 = \left(\frac{n_0 - n_1}{n_1 R}\right) r_0 + \left(\frac{n_0}{n_1}\right) \theta_0 \tag{1}$$

So, for a spherical surface, we can relate the refracted ray with the incident ray by a matrix relationship using equation (1) as follows:

$$\begin{bmatrix} r_1 \\ \theta_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{n_0 - n_1}{n_1 R} & \frac{n_0}{n_1} \end{bmatrix} \begin{bmatrix} r_0 \\ \theta_0 \end{bmatrix}$$

Thus the propagation of a ray through a spherical interface can be described by a 2 × 2 matrix generally called, in the literature, *ABCD matrix*. This can be generalized to many optical components [15] and to the case of reflection as follows:

$$\begin{bmatrix} r_1 \\ \theta_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} r_0 \\ \theta_0 \end{bmatrix}$$

If we have an optical system consisting of k optical components, then we can trace the input ray R_i through all optical components using composition of matrices of each optical component as follows:

$$R_o = (M_k.M_{k-1}....M_1).R_i \tag{2}$$

Simply, we can write $R_o = M_s R_i$ where $M_s = \prod_{i=k}^{1} M_i$. Here, R_o is the output ray and R_i is the input ray.

3 Formal Analysis Framework

The proposed framework, given in Figure 3, outlines the main idea behind the theorem-proving-based stability analysis of optical resonators. The grey shaded boxes in this figure show the key contributions of the paper that serve as the fundamental requirements for conducting formal stability analysis in a theorem prover. Like any system analysis tools, the inputs to this framework are the description of the optical resonator and geometric constraints, such as radius of curvature of mirrors and distance between different optical components. The

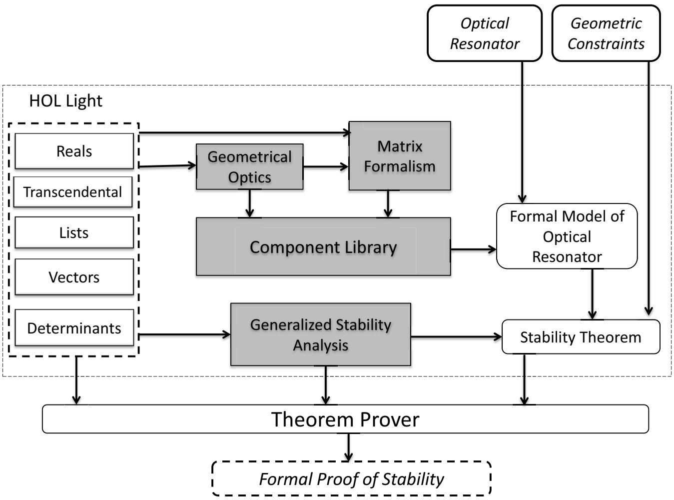

Fig. 3. Proposed Stability Analysis Framework for Optical Resonators

first step in conducting stability analysis of optical resonators using a theorem prover is to construct a formal model of the given resonator in higher-order logic.

For this purpose, the foremost requirement is the ability to formalize the underlying concepts of geometrical optics which includes the modeling of optical components and of the ray behaviour when it interacts with optical components. The second step in the proposed framework is to use the formalization of geometrical optics to formally derive the matrix formalism for geometrical components. This step requires the vector theory, which is already available as a part of multivariate analysis in HOL Light theorem prover. The third step to conduct formal stability analysis of optical resonators is to develop a library of frequently used optical components such as lenses, mirrors or crystals. Since such components are the basic blocks of optical systems, this library helps to formalize optical resonators. The next step is to formally define the stability of an optical resonator and verify some generalized stability theorems which are heavily dependent on matrix algebra within the HOL Light theorem prover. On top of that, one can finally state and prove the stability of an optical resonator in the theorem prover. The corresponding proof provides the output of the framework.

4 Formalization of Geometrical Optics

In order to fulfil the first requirement of the proposed stability analysis framework, we present the formalization of geometrical optics in this section. The formalization is two-fold: first, we model the geometry and physical parameters of an optical system; second, we model the physical behavior of a ray when it goes through an optical interface. Afterwards, we will be able to derive the ray-transfer matrices of the optical components, as explained in Section 2. We first define a type to describe optical systems:

Definition 1 (Optical Interface and System).

```
define_type "optical_interface = plane | spherical real"
define_type "interface_kind = transmitted | reflected"
new_type_abbrev("free_space",‘:real # real‘)
new_type_abbrev("optical_system",‘:(free_space # optical_interface #
                                   interface_kind) list # free_space‘)
```

An optical system is a list of free spaces and interfaces between them. A free space is represented by one real number for its refractive index and one for its width. Optical interfaces are characterized both by their shape (plane or spherical, as shown in Figure 4) and by the behavior of the ray when it goes through it (transmitted or reflected), thus yielding the two above types `‘:optical_interface‘` and `‘:interface_kind‘`. A spherical interface takes a real number representing its radius of curvature. A term of type `‘:free_space # optical_interface # interface_kind‘` is called an *optical component*. Note that this data type can easily be extended to many other optical components if needed.

A value of type `‘:free_space‘` does represent a real space only if the refractive index is greater than zero. In addition, in order to have a fixed order in the representation of an optical system, we impose that the distance of an optical interface relative to the previous interface is greater or equal to zero. We also need

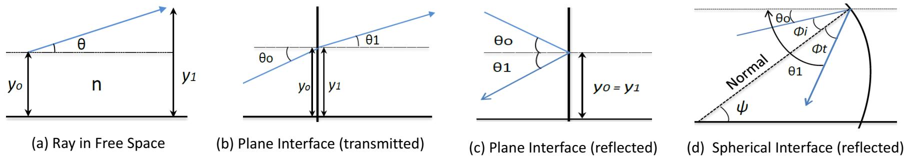

Fig. 4. Behavior of Ray at Different Interfaces

to assert the validity of a value of type `optical_interface` by ensuring that the radius of curvature of spherical interfaces is never equal to zero. These constraints are all packaged in a predicate `is_valid_optical_system os` which is true if and only if all the optical components of `os` satisfy the above requirements (the definition of this predicate is straightforward, see [16] for details).

We can now formalize the physical behaviour of a ray when it passes through an optical system. We only model the points where a ray hits an optical interface (instead of all the points constituting the ray). So it is sufficient to just provide the distance of the hitting point to the optical axis and the angle taken by the ray at that point. Consequently, we should have a list of such pairs (*distance*, *angle*) for every component of a system. In addition, the same information should be provided for the source of the ray. For the sake of simplicity, we define a type for a pair (*distance*, *angle*) as `ray_at_point`. This yields the following definition:

Definition 2 (Ray).

```
new_type_abbrev ("ray_at_point", `:real # real`)
new_type_abbrev ("ray", `:ray_at_point # ray_at_point #
                        (ray_at_point # ray_at_point) list`)
```

The first `ray_at_point` is the pair (*distance*, *angle*) for the source of the ray, the second one is the one after the first free space, and the list of `ray_at_point` represents the same information for all hitting points of an optical system. It is not necessarily the case that every value of type `ray` constitutes a valid ray, we thus constrain this type by using a predicate `is_valid_ray_in_system ray sys` which asserts that the value `ray` indeed represents a ray travelling in the system `sys` [16]. For example, Figure 4 provides a couple of situations which are formalized by `is_valid_ray_in_system ray sys`.

Now, as explained in Section 2, the behavior of a ray through an optical system can be conveniently expressed by matrices. In our formalism, the matrix corresponding to an optical system `os` is given by the function `system_composition os`. For the sake of conciseness, we do not provide the detailed definition of this function, which can be found in [16]. We then obtain the following essential result:

Theorem 1 (Ray-Transfer-Matrix for Optical System).

```
⊢ ∀ sys ray. is_valid_optical_system sys ∧
  is_valid_ray_in_system ray sys ⟹
  let (y0,θ0),(y1,θ1),rs = ray in
```

```
let yn,θn = last_ray_at_point ray in
[yn; θn] = system_composition sys * [y0; θ0]
```

$$\begin{bmatrix} y_n \\ \theta_n \end{bmatrix} = \texttt{system\_composition sys} * \begin{bmatrix} y_0 \\ \theta_0 \end{bmatrix}$$

where the function `last_ray_at_point` returns the last `ray_at_point` in system.

This concludes our formalization of geometrical optics and the verification of the generalized ray-transfer-matrix relationship (Theorem 1) of optical systems. The formal verification of the above important theorem reassures the correctness of our formal definitions related to optical systems. Now, we present the formalization of stability of an optical resonator and the verification of the generalized stability theorem in the following section.

5 Formalization of the Stability of Optical Resonators

Optical resonators are particular type of optical systems which are broadly classified as stable or unstable. One of the most interesting features of optical resonators is their diverse applications, e.g., stable resonators are used in the measurement of the refractive index of cancer cells [20], whereas unstable resonators are used in the laser oscillators for high energy applications [19]. Stability analysis identifies geometric constraints of the optical components which ensure that light remains inside the resonator (see Figure 5 (a)). In order to

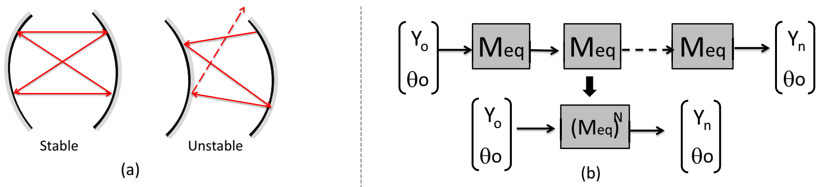

Fig. 5. (a) Types of Optical Resonators (b) ABCD Matrix After N Round-Trips

determine whether a given optical resonator is stable, we need to analyze the ray behaviour after many round trips. To model N round trips of light in the resonator, engineers usually "unfold" N times the resonator description, and compute the corresponding ray-transfer matrix. From the results presented in the previous section, it follows that it is equivalent to take the ray-transfer matrix corresponding to one round-trip and then raise it to the N^{th} power, as shown in Figure 5 (b). For an optical resonator to be stable, the distance of the ray from the optical axis and its orientation should remain bounded whatever the value of N. This is formalized as follows:

Definition 3 (Resonator Stability).

```
⊢ ∀ M. stable_optical_system M ⇔ (∀ X. ∃ Y. ∀ N.
abs((M mat_pow N) * X)$1 ≤ Y$1 ∧ abs((M mat_pow N) * X)$2 ≤ Y$2)
```

where `X` and `Y` are 2-dimensional vectors and `M` is a 2×2 matrix (intended to be the round-trip matrix of the resonator). The function `mat_pow` denotes the matrix power function and `V$i` denotes the i^{th} component of a vector `V`.

Proving that a given resonator satisfies the abstract condition of Definition 3 does not seem trivial at first. However, if the determinant of `M` is 1 (It means that the refractive index is the same at the input and output of the system. This is generally the case for optical systems encountered in practice), optics engineers have known for a long time that having $-1 < \frac{M_{11}+M_{22}}{2} < 1$ is sufficient to ensure that the stability condition holds. The obvious advantage of this criterion is that it is immediate to check. In order to prove this result, we can use Sylvester's Theorem [22,24], which states that for a matrix M$=\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ such that $\mid M \mid = 1$ and $-1 < \frac{A+D}{2} < 1$, the following holds:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^N = \frac{1}{\sin(\theta)} \begin{bmatrix} A\sin[N(\theta)] - \sin[(N-1)\theta] & B\sin[N(\theta)] \\ C\sin[N(\theta)] & D\sin[N(\theta)] - \sin[(N-1)\theta] \end{bmatrix}$$

where $\theta = cos^{-1}[\frac{(A+D)}{2}]$. This theorem ensures that stability holds under the considered assumptions: Indeed, N only occurs under a sine in the resulting matrix; since the sine itself is comprised between -1 and 1, it follows that the components of the matrix are obviously bounded, hence the stability. We formalize Sylvester's theorem as follows:

Theorem 2 (Sylvesters Theorem).

$$\vdash \forall\ \mathtt{N\ A\ B\ C\ D}.\ \begin{vmatrix} A & B \\ C & D \end{vmatrix} = 1 \wedge -1 < \frac{\mathtt{(A+D)}}{2} \wedge \frac{\mathtt{(A+D)}}{2} < 1 \Longrightarrow$$
$$\mathtt{let}\ \theta = \mathtt{acs}(\frac{\mathtt{(A+D)}}{2})\ \mathtt{in}$$
$$\begin{bmatrix} \mathtt{A} & \mathtt{B} \\ \mathtt{C} & \mathtt{D} \end{bmatrix}^{\mathtt{N}} = \frac{1}{\mathtt{sin}(\theta)} \begin{bmatrix} \mathtt{A} * \mathtt{sin}[\mathtt{N}(\theta)] - \mathtt{sin}[(\mathtt{N}-1)\theta] & \mathtt{B} * \mathtt{sin}[\mathtt{N}(\theta)] \\ \mathtt{C} * \mathtt{sin}[\mathtt{N}(\theta)] & \mathtt{D} * \mathtt{sin}[\mathtt{N}(\theta)] - \mathtt{sin}[(\mathtt{N}-1)\theta] \end{bmatrix}$$

We prove Theorem 2 by induction on N and using the fundamental properties of trigonometric functions, matrices and determinants. Now, we derive the generalized stability theorem for any ABCD matrix as follows:

Theorem 3 (Generalized Stability Theorem).

$$\vdash \forall\ \mathtt{A\ B\ C\ D}.\ \begin{vmatrix} A & B \\ C & D \end{vmatrix} = 1 \wedge -1 < \frac{\mathtt{(A+D)}}{2} \wedge \frac{\mathtt{(A+D)}}{2} < 1 \Longrightarrow$$
$$\mathtt{stable\_optical\_system}\ \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

The formal verification of Theorem 3 requires the formal definition of stability (Definition 3) and Sylvester's theorem along with some fundamental properties of vectors. It is important to note that our stability theorem is quite general and can be applied to any ABCD matrix which satisfies the required assumptions. This completes our formalization of stability and we present its practical effectiveness by analyzing Fabry Pérot resonator in the next section.

6 Application: Stability Analysis of Fabry Pérot Resonator

Nowadays, optical systems are becoming more and more popular due to their huge potential of application. In order to bring this technology to the market, a lot of research has been done toward the integration of low cost, low power and portable building blocks in optical systems. One of the most important such building blocks is the Fabry Pérot (FP) resonator [15]. Originally, this resonator was used as a high resolution interferometer in astrophysical applications. Recently, the Fabry Pérot resonator has been realized as a microelectromechanical (MEMS) tuned optical filter for applications in reconfigurable Wavelength Division Multiplexing [14]. The other important applications are in the measurement of refractive index of cancer cells [20] and optical bio-sensing devices [1].

Due to diverse applications of the FP resonators, different architectures have been proposed in the open literature. The main limitation of traditional designs is the instability of the resonators which prevents their use in many practical applications (e.g., refractometry for cancer cells). Recently, a state-of-the-art FP core architecture has been proposed which overcomes the limitations of existing FP resonators [10,9]. In the new design, cylindrical mirrors are combined with a fiber rod lens (FRL) inside the cavity, to focus the beam of light in both transverse planes as shown in Figure 6 (a). The fiber rod lens is used as light pipe which allows the transmission of light from one end to the other with relatively small leakage. Building a stable FP resonator requires the geometric constraints to be determined in terms of the radius of curvature of mirrors R and the free space propagation distance (d_{free_space}) using the stability analysis.

As a direct application of the framework developed in the previous sections, we present the stability analysis of FP resonator with fiber rod lens as described above. It is important to note that the design shown in Figure 6 (a), has a 3-dimensional structure. We can still apply the ray-transfer-matrix approach to analyze the stability by dividing the given architecture into two planes, i.e., XZ and YZ planes. Now, the stability problem becomes a couple of planar problems which are still valid since the ray focusing behaviours in both directions (XZ and YZ) are decoupled. This is merely a consequence of the decomposition of Euclidean space vectors into a basis. This can be seen in Figure 6 (b) and (c), where the resonator is divided into two cross-sections. In the following, we focus only on the analysis of the XZ plane, since the analysis in the YZ plane is fairly similar (the complete analysis can be found in the source code [17]).

In the XZ cross-section (Figure 6 (b)), the focusing is done by the curved mirrors. The fiber rod lens acts as a refracting slab with width d_f and refractive index n_f. The first step in the stability analysis, as described in our proposed framework is to construct a formal model of the given resonator in higher-order logic. A ray that makes a round-trip in the cavity undergoes (from left to right) first reflection in a curved mirror of radius R, propagation through free space of length d_x and refractive index 1, refraction from free space to fiber rod lens, propagation within fiber rod lens of length d_f and refractive index n_f, refraction from fiber rod lens to free space and again the propagation through free space

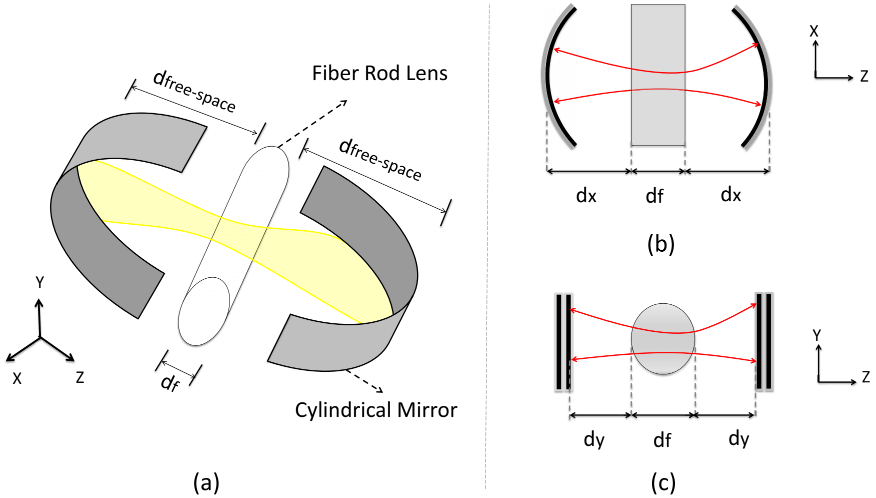

Fig. 6. Fabry Pérot (FP) Resonator with fiber rod lens (a) 3-Dimensional Resonator Design (b) Cross-Section view in the XZ Plane (c) Cross-Section view in the YZ Plane

of length d_x. Of course, the "return-trip" is symmetric. We formally model this system as follows:

Definition 4 (Formal Model of FP Resonator in XZ Plane).

```
⊢ ∀ R dx nf df. FP_XZ R dx df nf =
    ([(1,0),spherical R,reflected;(1,dx),plane,transmitted;
    (nf,df),plane,transmitted],1,dx)
```

Here, the pair `(1,0)` represents free space with refractive index 1 and null width. `FP_XZ` is a higher-order logic function which takes the parameters, radius of curvature of mirror (`R`), free space length (`dx`), length of fiber rod lens (`df`) and refractive index (`nf`). It returns an optical system (Definition 1) which corresponds to the formalization of a cavity with the corresponding input parameters. Next, we formally verify that the formal model of the cavity is valid under realistic geometric constraints, such as the fact that the refractive index (`nf`) and lengths of free space propagation (`dx` and `df`) should be greater than 0.

Theorem 4 (Validity of FP resonator in XZ Plane).

```
⊢ ∀ R dx df nf. R ≠ 0 ∧ 0 < dx ∧ 0 < df ∧ 0 < nf ⟹
    is_valid_optical_system (FP_XZ R dx df nf)
```

Next, we formally verify the equivalent matrix relationship of FP resonator in XZ plane using the formal definition of system composition.

Theorem 5 (Equivalent Matrix for FP resonator in XZ Plane).

```
⊢ ∀ R dx df nf. R ≠ 0 ∧ 0 < dx ∧ 0 < df ∧ 0 < nf ⟹
```

```
system_composition (FP_XZ R dx df nf) =
```

$$\begin{bmatrix} 1 - \frac{2 * (df + 2 * dx * nf)}{nf * R} & 2 * dx + \frac{df}{nf} \\ -\frac{2}{R} & 1 \end{bmatrix}$$

The verification of this theorem mainly involves the matrix algebra and some arithmetic reasoning. The following result is then easy to prove by making use of the results already obtained in our framework:

Theorem 6 (Ray-Transfer-Matrix Model in XZ plane).

$\vdash \forall$ R dx df nf. R $\neq$ 0 $\wedge$ 0 < dx $\wedge$ 0 < df $\wedge$ 0 < nf $\Longrightarrow$
($\forall$ ray.is_valid_ray_in_system ray (FP_XZ R dx df nf)
$\Longrightarrow$ (let (y_0,θ_0),(y1,theta1),rs = ray in
(y_n,θ_n) = last_single_ray ray in
vector [y_n;θ_n] = system_composition (FP_XZ R dx df nf) *
vector [y_0;θ_0]))

where `last_single_ray` is a function that takes a `ray` as input and returns the last pair (distance from the optical axis y and the orientation θ) of that `ray`.

To this point, we have formally developed the model of the FP resonator in the XZ plane and also verified important properties such as the validity of the model and the ray-transfer-matrix relationship. Now, we are in a position to formally verify the stability of the FP resonator in the XZ plane, which is the final step.

Theorem 7 (Stability in XZ plane).

$\vdash \forall$ R dx df nf. R $\neq$ 0 $\wedge$ 0 < dx $\wedge$ 0 < df $\wedge$ 0 < nf
0 < $\frac{2*dx+\frac{df}{nf}}{R}$ $\wedge$ $\frac{2*dx+\frac{df}{nf}}{R}$ < 2 $\Longrightarrow$ stable_optical_system
(system_composition (FP_XZ R dx df nf))

The first four assumptions just ensure the validity of the model description. The two following ones provide the intended stability criteria. The formal verification of Theorem 7 requires Theorem 5 and Theorem 3 along with some fundamental properties of matrices and arithmetic reasoning.

Similarly, we can model and verify the validity of the FP resonator in YZ plane by performing the above mentioned steps. For the sake of conciseness, we only present the stability theorem in YZ plane as follows:

Theorem 8 (Stability in YZ plane).

$\vdash \forall$ dy df nf. 0 < dy $\wedge$ 0 < df $\wedge$ 0 < nf
0 < 1 $-$ $\frac{2}{nf}$ + $(4 * \frac{dy}{df}) * (1 - \frac{1}{nf})$ $\wedge$ 1 $-$ $\frac{2}{nf}$ + $(4 * \frac{dy}{df}) * (1 - \frac{1}{nf})$ < 1
$\Longrightarrow$ stable_optical_system (system_composition (FP_YZ dy df nf))

The first three assumptions just ensure the validity of the model description. The two following ones provide the intended stability criteria.

It is important to note that for the FP resonator with fiber rod lens, we have two sets of stability constraints, i.e., in the XZ plane (Theorem 7) and in the YZ plane (Theorem 8). Consequently, the resonator can be stable in one plane and unstable in the other. Therefore, in practice, the criteria of Theorem 7 and 8 should both be satisfied.

This completes our formal stability analysis of the FP resonator with fiber rod lens, which clearly demonstrates the effectiveness of the proposed theorem proving based stability analysis framework. The above formal analysis allowed us to find some discrepancy in the paper-and-pencil based proof approach presented in [10]. Particularly, the order of matrix multiplication in Equations (16) and (24) in [10] should be reversed, so as to obtain correct stability constraints. Due to the formal nature of the model and inherent soundness of higher-order logic theorem proving, we have been able to verify the stability of Fabry Pérot (FP) resonator with fiber rod lens with an unrivaled accuracy. This improved accuracy comes at the cost of the time and effort spent, while formalizing the underlying theory of geometrical optics and resonator stability. But, the availability of such a formalized infrastructure significantly reduces the time required to analyze the Fabry Pérot (FP) resonator with fiber rod lens. Moreover, we automatized parts of the verification task by introducing new tactics, e.g., `VALID_OPTICAL_SYSTEM_TAC`, which automatically verifies the validity of a given optical system. We also formally analyzed a couple of other important resonator architectures such as FP resonator with curved mirrors and Z-shaped resonator. Our HOL Light developments of geometrical optics, Fabry Pérot (FP) resonators and Z-shaped resonator are available for download [17] and thus can be used by other researchers and optical engineers working in industry to conduct the formal stability analysis of their optical resonators.

7 Conclusion

In this paper, we report a novel application of formal methods in the stability analysis of optical resonators which is mainly based on geometrical optics. We provided a brief introduction of the current state-of-the-art and highlighted their limitations. Next, we presented an overview of geometrical optics followed by some highlights of our higher-order logic formalization. In order to show the practical effectiveness of our proposed framework, we presented the formal stability analysis of Fabry Pérot (FP) resonator with fiber rod lens. Note that this application is not a simple toy example but an advanced system which has been published only recently. In fact, we were able to identify some discrepancy in the paper-and-pencil based stability analysis presented in [10]. Catching this problem in paper-and-pencil based proofs clearly indicates the usefulness of using higher-order-logic theorem proving for the stability analysis of optical resonators. To the best of our knowledge, this is the first time that formal approach has been applied for the stability analysis of optical resonators.

The rigor of formal verification allows to go beyond what is traditionally done by optics engineers. For instance, during our formalization, we have identified that the paraxial approximation is not taken into account rigorously in traditional techniques. However, theorem proving provides the required mathematical background to tackle this precisely. This is one of our essential future work.

We also plan to automatize the verification of optical resonators' stability by developing dedicated conversions and tactics that would compute automatically the required matrix products and check that the resulting matrices indeed satisfy the conditions given by Sylvester's theorem. In the future, we also plan to extend this work in order to obtain an extensive library of verified optical components, along with the formalization of Gaussian beams, which would allow the formal analysis of resonator modes [19]. We also plan to package our HOL Light formalization in a GUI, so that it can be used by non-formal methods community in industry for the analysis of practical resonators and in academia for teaching and research purposes.

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