Formal analysis of electromagnetic optics

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ABSTRACT

Optical systems are increasingly being used in safety-critical applications. Due to the complexity and sensitivity of optical systems, their verification raises many challenges for engineers. Traditionally, the analysis of such systems has been carried out by paper-and-pencil based proofs and numerical computations. However, these techniques cannot provide accurate results due to the risk of human error and inherent approximations of numerical algorithms. In order to overcome these limitations, we propose to use theorem proving (i.e., a computer-based technique that allows to express mathematical expressions and reason about their correctness by taking into account all the details of mathematical reasoning) as a complementary approach to improve optical system analysis. This paper provides a higher-order logic (a language used to express mathematical theories) formalization of electromagnetic optics in the HOL Light theorem prover. In order to demonstrate the practical effectiveness of our approach, we present the analysis of resonant cavity enhanced photonic devices.

Keywords: Electromagnetic optics, formal verification, theorem proving, higher-order logic, HOL Light

1. INTRODUCTION

Optical systems are being increasingly used in many safety-critical domains, ranging from microsystems and telecommunication to medicine, laser industry, aerospace, and military technologies. Verification of such systems is one of the most critical and time consuming steps in system development and enhancing the performance of the verification process has a great impact on the key business drivers of quality, schedule and cost. Due to the complexity and sensitivity of optical systems, the validation of system models and verification of the system properties raise many challenges for engineers.

The verification of an optical system is generally achieved by combining various means. The most basic one is the actual manufacturing of a prototype that can then be tested. However, there are instances where this costly process cannot be accomplished, e.g., for large optical space borne systems, it is almost impossible to perfectly reproduce space conditions on the ground. Therefore, engineers try as much as they can to detect faults in a design before resorting to testing. This requires developing a mathematical model of the system and then analyzing it. Such a model is based on various theories of physics depending on the system properties that need to be verified.

Once the system modelling is finalized, the model has to be analyzed to ensure that it exhibits the desired optical system properties. The traditional method for analyzing an optical system model, is the paper-and-pencil approach. However, as system behaviours become more complex, this approach can become extremely cumbersome and more prone to human error. Many examples of erroneous paper-and-pencil based proofs are available in the open literature, for example we can refer to a study on planar waveguides containing chiral nihility metamaterial[1] and its identification and correction report.[2] A natural improvement on this method is to use numerical approaches and computer-aided-design algorithms and the associated softwares. In this context, the analysis is mostly done using computer simulation[3] and Computer Algebra Systems ("CASs").[4] Although these techniques make the analysis less cumbersome – and thus less error-prone, due to the nature of numerical algorithms and large set of unverified simplification algorithms,[5] one should not rely *solely* on these traditional techniques, especially when used in safety-critical areas (e.g., laser surgery), where system flaws may result in the loss of human lives.

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Given the enormous usage of optical systems in safety-critical applications, ranging from astronomy[6] to the refractive index measurement of cancer cells,[7] their precise analysis is a dire need. Therefore, we propose a higher-order-logic theorem proving based framework for the accurate analysis of electromagnetic theory based optical system models.

The formal analysis of optical systems has gained some interest from the community recently (e.g.,[8–10]). For the first time, in 2009, Hasan et al.[8] presented the idea of formalizing optics using theorem proving by formal analysis of a planar waveguide. However, the authors considered the electromagnetic field equations of the planar waveguide as real functions, which make the formal definition of electromagnetic fields not general enough to address many applications. The next significant work on formal analysis of optics, done by Siddique et al.,[9,10] focused on formal analysis of geometrical optics addressing the stability analysis of resonators. However, the accuracy of geometrical optics is limited to find basic properties of optical systems, such as approximate image and object positions and magnifications, when the wavelength is small compared to other geometrical features of the optical system. This condition is clearly not adequate for studying many optical systems of present day with dimensions in the order of an optical wavelength or less. Recently, Khan-Afshar et al.[11] reported their effort on formalization of optical interconnects based on electromagnetic optics. However, the work is focused on formalizing individual components without providing a foundation to extend the work. In 2013, Gay and Nagarajan reported formal modelling and analysis of quantum systems[12] using three different approaches, i.e., behavioural equivalence in process calculus, model-checking, and equivalence checking. The authors reason about why formal techniques are necessary for the analysis of large-scale systems that combine quantum and classical components, e.g., quantum optics, however, their approach addresses system level designs. We also can refer to the work of Mahmoud et al.[13,14] in quantum optics where formal verification of beam splitters and quantum flip gates are presented, respectively. To the best of our knowledge, the most comprehensive work on formalization of optical systems is a general framework presented in 2013, by Khan-Afshar et al.,[15] to use theorem proving as a complement to computational and numerical approaches to improve analysis of optical system models, in ray, electromagnetic and quantum optics.

In this paper, we provide a generic framework (solely focused) on formalization of electromagnetic optics. The paper also presents the formalization of light as an electromagnetic wave and the formal verification of some of its key properties, like the law of reflection, Snell's law, Fresnel's equations, which are the foremost foundations of the physical-optics. We have used the HOL Light theorem prover for this development mainly because of the availability of the required mathematical theories in its library. The rest of the paper is organized as follows: in Section 2, we briefly describe the idea of verification using formal methods focusing on theorem proving, and the physical foundations of optics, focusing on electromagnetism. Section 3 presents our framework on verification of optical systems. Section 4 details the higher-order-logic formalization of the fundamentals of the proposed approach. Section 5 illustrates the practical use of our framework by describing the analysis of the Fabry-Pérot resonator; an essential structure of many complex optoelectronic systems, like lasers and photo-detectors. Finally, Section 6 concludes the paper.

2. PRELIMINARIES

In this section, we first give a brief introduction to formal methods, in general, and higher-order-logic theorem proving, in particular. The idea is to reason about why we chose theorem proving to formalize electromagnetic optics. Then, by a brief introduction to electromagnetic optics, we present what we need to formalize as an infrastructure to pave the way to the formal analysis of electromagnetic optics.

2.1 Formal Verification and Theorem Proving

Formal verification is the process employed to prove that a system is designed in accordance to its requirements using computer-based mathematical reasoning. The formal description of the requirements is referred to as formal specification and the system design is referred as formal implementation. In theory, formal verification can be done at all levels of abstraction, as long as, formal specification and formal implementation can be extracted and there exists a mathematical-relation which can be established between the two descriptions. There are two main approaches to formal verification: *automated state-space exploration* and *theorem proving*.

The main state-space exploration methods are "model checking" and "equivalence checking".[16] In "model checking", the system model (implementation) is defined as a finite state machine, and specifications are written in temporal logic. Then, for each property written in temporal logic, an exhaustive search through the state space of the system takes place to determine if the property holds. In case the property does not hold, a counter example describing the failure point(s) is generated. Then the design can be corrected and reverified. In "equivalence checking", the output signals of two different models of the designs are compared for a given set of input conditions. Correctness of the method relies on the exploration and comparison of the reachable state spaces. In general, the main advantage of state exploration methods is their automation, while the main drawback is the state explosion problem which is the principle limiting factor of this technology. The efficiency of state exploration methods depends heavily on the size of the reachable state space. As the size of a state machine describing the system model increases, exhaustive search becomes practically infeasible. In optical systems, the continuous nature of the analysis prevents the system from being abstracted within a finite state machine without loosing accuracy.

Theorem proving is an approach where both the system and its desired properties are expressed as formulae in some mathematical logic. This logic is defined by a formal system, called proof system or calculus, which defines a set of axioms and a set of inference rules. Theorem proving is the process of deriving formal proofs from the basic axioms and possibly intermediate lemmas using inference rules. The axioms are usually "elementar" in the sense that they capture the basic properties of the logic's operators. Many theorem-proving systems have been implemented and used for all kinds of verification problems. These systems are distinguished by, among other aspects, the underlying mathematical *logic*, the way automatic decision procedures are integrated into the system and the user interface.

In general, a *logic* provides a (formal) language to express mathematical facts, and a definition of what a *true sentence* is in this language. The most basic kind of logic is the *propositional logic* (also called *boolean logic*), which only allows sentences formed by propositional variables and boolean connectives: *and* ($\wedge$), *or* ($\vee$), *not* ($\neg$), *implies* ($\Rightarrow$) and equality ($=$) connectives. For instance, $(\mathtt{A} \Rightarrow \mathtt{B}) \wedge (\mathtt{B} \Rightarrow \mathtt{C}) \Rightarrow (\mathtt{A} \Rightarrow \mathtt{C})$ is a sentence of propositional logic. In addition, one can easily see that it is a *true* sentence (using the transitivity of implication).

Only the overall structure of mathematical sentences can be expressed in propositional logic and one lacks the ability to talk about objects and their properties. This problem is answered by first-order logic that introduces *terms* (which formalize the notion of "object") and *predicates* (which formalize the notion of "property of an object"). Terms are built inductively from constants and functions, e.g., the set of natural numbers is built from the constant $\mathtt{0}$ and the function $\mathtt{SUC}$, hence, $\mathtt{1}$ is represented by $\mathtt{SUC}(\mathtt{0})$, $\mathtt{2}$ by $\mathtt{SUC}(\mathtt{SUC}(\mathtt{0}))$, etc. Being an even or a prime number are then properties of natural numbers that can be represented by predicates. First-order logic thus allows to write sentences like $\mathtt{Even}(\mathtt{0})$ or $\mathtt{Prime}(\mathtt{SUC}(\mathtt{SUC}(\mathtt{0}))$. In order to get closer to the usual mathematical language, first-order logic also introduces the notion of a *variable*. Finally, sentences with variables are not complete if we cannot specify how variables should be interpreted, so two new ways of building a sentence are added to the language by using *for all* ($\forall$) and *there exists* ($\exists$) (called *quantifiers*): e.g., "$\forall \mathtt{x}.\ \mathtt{Even}(\mathtt{x}) \ \Rightarrow\ \mathtt{Even}(\mathtt{SUC}(\mathtt{SUC}(\mathtt{x}))$" or "$\exists \mathtt{x}.\ \mathtt{Prime}(\mathtt{x})$".

First-order logic does not permit quantifying over predicates. For instance, it is impossible to express the induction principle for natural numbers: $\forall \mathtt{P}.\mathtt{P}(\mathtt{0}) \wedge (\forall \mathtt{n}.\mathtt{P}(\mathtt{n}) \Rightarrow \mathtt{P}(\mathtt{SUC}(\mathtt{n}))) \Rightarrow \forall \mathtt{n}.\mathtt{P}(\mathtt{n})$ since $\forall$ can only be applied to variables and not to predicates. Higher-order logic provides this feature and thus, in comparison to the aforementioned logic is stronger to represent mathematical theories. Theoretically, any system that can be expressed in a closed mathematical form can be defined in higher-order logic due to highly expressive nature of this logic.

Given a logic, the most frequent problem is to try to determine whether a given sentence is true or not. This is done by considering a set of *axioms*, i.e., basic sentences that are assumed to be true (e.g., $\mathtt{P} \vee \neg \mathtt{P}$), and *inference rules*, i.e., rules that allow to derive the truth of a sentence depending upon the truth of other sentences (e.g., if $\mathtt{P}$ and $\mathtt{Q}$ are true sentences, then $\mathtt{P} \wedge \mathtt{Q}$ is a true sentence). Using axioms and inference rules, one can thus *prove* or disprove logical sentences. This idea is at the principle core of theorem proving: the language definition, the axioms and inference rules can be implemented in the theorem prover. This allows the user to write down mathematical sentences inside the theorem prover, and then to prove them *using only* the axioms and inference rules provided by the theorem prover. This latter point is essential since, assuming there exists no inconsistencies in the foundations of the theorem prover, it ensures that no unsound reasoning step can be used to prove a theorem. This guarantees that any sentence, which is proved in a theorem prover, is indeed true. Thus, theorem proving is a powerful verification technique which can provide a unifying framework for various verification tasks at different hierarchical levels. However, the task of proving complex theorems requires expertise in both formalization and the application in hand, in addition to great effort and creativity on the part of the verifier.

2.2 Electromagnetic Theory of Light

In the electromagnetic theory, light is described by the same principles that govern all forms of electromagnetic radiations. An electromagnetic radiation is composed of an electric and a magnetic field. The general definition of a field is "a physical quantity associated with each point of space-time". Considering electromagnetic fields ("EMFs"), the "physical quantity" consists of a 3-dimensional vector for the electric and the magnetic field. Consequently, both those fields are defined as vector functions $\vec{E}(\vec{r},t)$ and $\vec{H}(\vec{r},t)$, respectively, where $\vec{r}$ is the position and t is the time. These functions are related by the well-known Maxwell equations[17] (e.g., in differential form):

$$\begin{aligned}
\nabla \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\
\nabla \times \vec{H} &= \vec{J} + \frac{\partial \vec{D}}{\partial t} \\
\nabla \cdot \vec{D} &= \rho \\
\nabla \cdot \vec{B} &= 0
\end{aligned} \tag{1}$$

with their associated constitutive equations,

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P} = \varepsilon \vec{E} \qquad \text{and} \qquad \vec{B} = \mu_0(\vec{H} + \vec{M}) = \mu \vec{H}$$

where $\vec{D}$ and $\vec{B}$ are the electric and magnetic flux density, respectively, $\vec{J}$ the electric current density, ρ the electric charge density, and $\nabla\times$ and $\nabla\cdot$ denote the curl operation and divergence, respectively. The parameters ε and μ represent the permittivity and permeability in the medium, and ε_0 and μ_0 are permittivity and permeability in free space, respectively. The vector fields $\vec{P}$ and $\vec{M}$ represent the polarization and the magnetization density, which are measures of the response of the medium to the electric and magnetic fields, respectively.[18] Once the medium is known, an equation relating $\vec{P}$ and $\vec{E}$, and another relating $\vec{M}$ and $\vec{H}$ is established. When substituted in Maxwell equations, the set of partial differential Equations (1) will be simplified governing only the two vector fields $\vec{E}$ and $\vec{H}$. Therefore, to describe electromagnetic waves in a medium, it would be enough to describe the *medium*, and the *electromagnetic fields*: $\vec{E}$ and $\vec{H}$.

3. FORMAL ANALYSIS FRAMEWORK

The proposed approach for the formal analysis of optical system models is depicted in Figure 1. Just like any system analysis problem, the inputs to the proposed analysis framework are an informal description of the optical system model and an informal specification of the desired properties of the system. The first step towards formal analysis of a system is to have both the specification of the system (in terms of properties), and implementation of the system *formally* described (in our case, using higher-order logic). The next step is to formally verify in a theorem prover that the implementation implies all the properties extracted from the specifications. Obviously, a mathematical correlation must exist between the formal specification and the formal model.

The formalization of ray, wave, electromagnetic, and quantum optics play a vital role in both of these steps. Firstly, they provide the means to describe the specification and system model formally. Secondly, they also provide the formal reasoning support for verifying the system properties. Since our focus, in this paper, is on the electromagnetic theoretic analysis of optical systems, we proceed with the formalization of the system behaviour by formalizing its EMF and medium aspects. The EMF aspects of an optical system are usually described mathematically using complex vectors. Whereas the medium aspects are mathematically expressed using euclidean geometry. As it can be depicted in Figure 1, we need to have both concepts of complex vectors

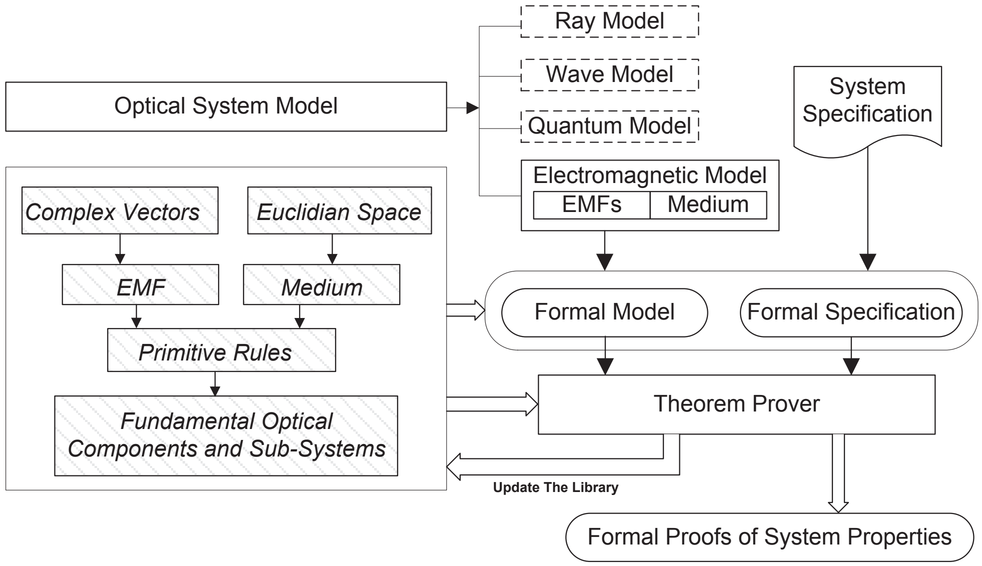

Figure 1. Formal Analysis Framework

and Euclidean space formalized in our libraries. Next, the properties of the system have to be verified within the sound core of theorem prover. This can of course be done from scratch by using only the inference rules of higher-order logic. But some fundamental results are always used, irrespective of the optical component that we want to verify, e.g., the law of Reflection, Snell's law, or Fresnel equations. So we propose to prove these foundations once and for all in order to make the verification of new components easier. This yields a "library of primitive rules of optics".

Optical systems are usually composed of some commonly used sub-systems like resonators or waveguides. Therefore, we also propose to formalize such often-used structures so that complex optical systems can be modelled and analyzed easily in a hierarchical manner. The fact that our formalization starts from the low-level roots of optics not only allows us to formalize these often-used structures, but also provides the ability to define new structures when needed. This makes our framework both easy to use in standard cases and flexible when we need to get out of the standards. Note that new formalized structures can be added to the library of components and sub-systems in order to be used without enduring the pain of formalizing them again. This fact is illustrated in Figure 1 by the feedback between the optic system analysis libraries and the theorem prover.

4. FORMALIZATION OF OPTICS FOUNDATIONS

The HOL Light theorem prover provides a formalization of complex numbers, real vectors, complex vectors,[19] and euclidean geometry. Using these libraries, in this section, the higher-order-logic formalization of the electromagnetic model of light wave and the formal verification of some primitive laws of optics are presented. These foundations play a vital role in analyzing optical systems based on the proposed framework of Figure 1.

4.1 Formalization of the Electromagnetic Model

As explained in Section 2.2, the electromagnetic theory considers light as an electromagnetic field ("EMF"). Thus, we first need to define a field. The general definition of a field is "a physical quantity associated with each point of space-time". Points of space are represented by 3-dimensional real vectors, so we define the type `point` as an abbreviation for the type $\texttt{real}^3$ ($\texttt{A}^\texttt{N}$ is the HOL Light library built-in type for vectors of size `N` whose components are of type `A`[20]). Instants of time are considered as real so the type `time` is a synonymous for the type `real` in our formalization. Finally, the "physical quantity" is formally defined as a 3-dimensional complex vector. Consequently, the type `field` (either magnetic or electric) is defined as $\texttt{point} \rightarrow \texttt{time} \rightarrow \texttt{complex}^3$. Then, since an EMF is composed of an electric and a magnetic field, we define the type `emf` to represent $\texttt{point} \rightarrow \texttt{time} \rightarrow \texttt{complex}^3 \times \texttt{complex}^3$.

A very general expression of an EMF is $\vec{U}(\vec{r},t) = \vec{a}(\vec{r})e^{j\phi(\vec{r})}e^{j\omega t}$, where $\vec{U}$ can be either the electric or magnetic field at point $\vec{r}$ and time t. We call $\vec{a}(\vec{r})$ the *amplitude* of the field and $\phi(\vec{r})$ its *phase*. Note that we consider only monochromatic waves, i.e., waves with only one *frequency* ω. This is a classic simplification which is not a restriction since any polychromatic wave can be decomposed into a sum of monochromatic ones.

Here, we focus on monochromatic *plane* waves, where the phase $\phi(\vec{r})$ has the form $-\vec{k}\cdot\vec{r}$, defined using the dot product between real vectors. We call $\vec{k}$ the *wavevector* of the wave; intuitively, this vector represents the propagation direction of the wave. This yields the following definition*:

Definition 4.1 (Plane wave).

$$\vdash_{\texttt{def}} \ \texttt{plane\_wave}\ (\texttt{k} : \mathbb{R}^3)\ (\omega : \mathbb{R})\ (\texttt{E} : \mathbb{C}^3)\ (\texttt{H} : \mathbb{C}^3) : \ \texttt{emf}$$
$$= \ \lambda(\texttt{r} : \texttt{point})\ (\texttt{t} : \texttt{time}).\ \ (\texttt{e}^{-\texttt{j}(\texttt{k}\cdot\texttt{r}-\omega\texttt{t})}\texttt{E}, \texttt{e}^{-\texttt{j}(\texttt{k}\cdot\texttt{r}-\omega\texttt{t})}\texttt{H})$$

where $\texttt{j}$ denotes $\sqrt{-1}$. Note that, although complex numbers are already defined in HOL Light,[21] we had to develop our own library of *complex vectors*[19] in order to define operations like addition, multiplication by a scalar or dot product for such vectors. In addition to Definition 4.1, we define the helper predicates and the functions `is_plane_wave`, `k_of_w`, ω`_of_w`, `e_of_w`, and `h_of_w` such that:

$$\forall \texttt{emf}.\ \ \texttt{is\_plane\_wave emf}\ \Leftrightarrow\ \ \texttt{emf} = \texttt{plane\_wave}\ (\texttt{k\_of\_w emf})(\omega\texttt{\_of\_w emf})(\texttt{e\_of\_w emf})\ (\texttt{h\_of\_w emf})$$

When a light wave passes through a medium, its behaviour is governed by different characteristics of the medium. The *refractive index* is the most dominant among these characteristics and thus we have used the data type `medium` to represent the medium with its refractive index, which is a real number. Most of the study of an optical device deals with the passing of light from one medium to another. So our basic system of study is the *interface* between two mediums. In general, such an interface can have any shape, but, most of the time, a plane interface is used, as shown in Figure 2.

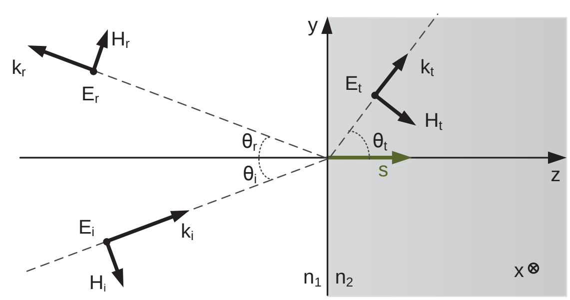

Figure 2. Plane Interface between Two Mediums

So we define the type `interface` as $\texttt{medium} \times \texttt{medium} \times \texttt{plane} \times \texttt{real}^3$, i.e., two mediums, a plane (defined as a set of points of space), and an orthonormal vector to the plane, indicating which medium is on which side of the plane.

Another useful consequence of Maxwell equations[18] is that the projection of the electric and magnetic fields shall be equal on both sides of the interface plane. This can be formally expressed by saying that the cross product between those fields and the normal to the surface shall be equal:

Definition 4.2 (Boundary conditions).

$$\vdash_{\texttt{def}} \ \texttt{boundary\_conditions emf}_1\ \texttt{emf}_2\ \texttt{n p t}\ \Leftrightarrow$$
$$\texttt{n} \times \texttt{e\_of\_emf emf}_1\ \texttt{p t} = \texttt{n} \times \texttt{e\_of\_emf emf}_2\ \texttt{p t}\ \wedge\ \texttt{n} \times \texttt{h\_of\_emf emf}_1\ \texttt{p t} = \texttt{n} \times \texttt{h\_of\_emf emf}_2\ \texttt{p t}$$

*From now on, all HOL Light statements will be written by mixing HOL Light script notations and pure mathematical notations in order to improve readability. Also $\mathbb{R}$ and $\mathbb{C}$ indicate the types real and complex, respectively.

where $\times$ denotes the *complex* cross product, and `e_of_emf` and `h_of_emf` are helper functions returning the electric and magnetic field components of an EMF.

We now have all the definitions that are required to reason over electromagnetic plane waves and the interface between two mediums. This reasoning is classically simplified by decomposing EMFs into two orthogonal EMFs with orientation towards the normal of interface plane. These two orthogonal EMFs are considered as two modes of any EMF and called *TE mode*, for "transverse electric", and *TM mode*, for "transverse magnetic". In the TE Mode, all electric fields are collinear and perpendicular to the normal of interface, while in the TM mode, all magnetic fields are collinear and perpendicular to the normal of interface. Definition 4.3 ensures aforementioned conditions in the TE mode.

DEFINITION 4.3 (TE MODE).
$\vdash_{\mathtt{def}}$ $\mathtt{te\_mode}\ (\mathtt{i} : \mathtt{interface})\ \mathtt{emf_i}\ \mathtt{emf_r}\ \mathtt{emf_t}\ \Leftrightarrow$
$\exists \mathtt{v}.\ \mathtt{orthogonal}\ \mathtt{v}\ (\mathtt{normal\_of\_interface}\ \ \mathtt{i}) \wedge \mathtt{norm}\ \mathtt{v}\ =\ 1\ \wedge$
$(\forall \mathtt{r}\ \mathtt{t}.\ \ \mathtt{vcollinear}\ (\mathtt{e\_of\_emf}\ \mathtt{emf_i}\ \mathtt{r}\ \mathtt{t})\ \mathtt{v}\ \wedge\ \mathtt{vcollinear}\ (\mathtt{e\_of\_emf}\ \mathtt{emf_r}\ \mathtt{r}\ \mathtt{t})\ \mathtt{v} \wedge$
$\quad\quad \mathtt{vcollinear}\ (\mathtt{e\_of\_emf}\ \mathtt{emf_t}\ \mathtt{r}\ \mathtt{t})\ \mathtt{v})$

where the functions `vcollinear` ensures collinearity between two vectors. The same definition is derived for the TM mode. Any theorem or definition related to one mode has a counterpart in the other. Since any EMF can be defined as the superposition of its TE and TM modes, if a property is formalized in one mode, it has to be verified in the other mode. Here, we focus on the TE mode.

4.2 Formalization of Primitive Rules of Optics

Now that the notions of EMF and interface have been formalized, we can prove some basic properties of optics that constitute the foundations to verify any optical system. We call them "primitives". Most of these primitives impose some particular constraints on the waves, for instance, some parameters must be positive, or non-null. One of the major advantages of theorem proving over other analytical methods is that these constraints are explicitly provided in the hypotheses of the corresponding theorems. This way, these theorems can only be applied if the corresponding constraints are ensured. As already explained, the study of an optical component mostly deals with the behaviour of light when it passes from one medium to another. Thus, we first formalize the simple case of a plane interface between two mediums, in the presence of a plane wave, shown in Figure 2, with the following predicate:

DEFINITION 4.4 (CHECKING IF THE WAVE IS A PLANE WAVE AT PLANE INTERFACE.).
$\vdash_{\mathtt{def}}$ $\mathtt{is\_plane\_wave\_at\_int}\ \mathtt{i}\ \mathtt{emf_i}\ \mathtt{emf_r}\ \mathtt{emf_t}\ \Leftrightarrow$
$\mathtt{is\_valid\_interface}\ \mathtt{i}\ \wedge\ \mathtt{non\_null}\ \mathtt{emf_i} \wedge$
$\mathtt{is\_plane\_wave}\ \mathtt{emf_i}\ \wedge\ \mathtt{is\_plane\_wave}\ \mathtt{emf_r}\ \wedge\ \mathtt{is\_plane\_wave}\ \mathtt{emf_t} \wedge$
$(\mathtt{let}\ (\mathtt{n_1}, \mathtt{n_2}, \mathtt{p}, \mathtt{n}) = \mathtt{i}\ \mathtt{in}\ \forall \mathtt{p}.\ \ \mathtt{t}\ \mathtt{is\_in\_plane}\ \mathtt{pt}\ \mathtt{p} \Rightarrow$
$\quad\quad \forall \mathtt{t}.\ \ \mathtt{boundary\_conditions}\ (\mathtt{emf_i} + \mathtt{emf_r})\ \mathtt{emf_t}\ \mathtt{n}\ \mathtt{pt}\ \mathtt{t}) \wedge$
$(\mathtt{let}\ (\mathtt{k_i}, \mathtt{k_r}, \mathtt{k_t}) = \mathtt{map\_trpl}\ \mathtt{k\_of\_w}\ (\mathtt{emf_i}, \mathtt{emf_r}, \mathtt{emf_t})\ \mathtt{in}$
$0 \leq (\mathtt{k_i} \cdot \mathtt{n})\ \wedge\ (\mathtt{k_r} \cdot \mathtt{n}) \leq 0\ \wedge\ 0 \leq (\mathtt{k_t} \cdot \mathtt{n})\ \wedge\ \exists \mathtt{k_0}.\ \ \ \mathtt{norm}\ \mathtt{k_i} = \mathtt{norm}\ \mathtt{k_r} = \mathtt{k_0 n_1}\ \wedge\ \mathtt{norm}\ \mathtt{k_t} = \mathtt{k_0 n_2}) \wedge$
$\mathtt{let}\ \mathtt{emf\_in\_med}\ = \lambda \mathtt{emf}\ \mathtt{n}.\ \ \mathtt{h\_of\_w}\ \mathtt{emf} = \frac{1}{\eta_0 \mathtt{k_0}}(\mathtt{k\_of\_w}\ \mathtt{emf}) \times (\mathtt{e\_of\_w}\ \mathtt{emf}))\mathtt{in}$
$\mathtt{emf\_in\_med}\ \mathtt{emf_i}\ \mathtt{n_1}\ \wedge\ \mathtt{emf\_in\_med}\ \mathtt{emf_r}\ \mathtt{n_1}\ \wedge\ \mathtt{emf\_in\_med}\ \mathtt{emf_t}\ \mathtt{n_2}$

where $\mathtt{map\_trpl}\ \mathtt{f}\ (\mathtt{x}, \mathtt{y}, \mathtt{z}) = (\mathtt{f}\ \mathtt{x}, \mathtt{f}\ \mathtt{y}, \mathtt{f}\ \mathtt{z})$ and η_0 is the impedance of vacuum, a physical constant relating magnitudes of electric and magnitude fields of electromagnetic radiation travelling through vacuum. The predicate of Definition 4.4 takes an interface `i` and three EMFs $\mathtt{emf_i}$, $\mathtt{emf_r}$, and $\mathtt{emf_t}$, intended to represent the incident wave, the reflected wave, and the transmitted wave, respectively. When `is_plane_wave_at_int` holds, it first ensures that the arguments are well formed, i.e., `i` is a valid interface and the three input fields are plane waves. It also ensures that the reflected wave exists by asserting that its electric field is non-null (both electric and magnetic fields of an EMF are not null) and goes from medium 1 to medium 2, and that the reflected and transmitted waves go in the opposite and same direction, respectively. These conditions are expressed by using the dot product of the wavevectors to the normal of the interface plane. Moreover, Definition 4.4 also ensures that the boundary conditions shall hold at every point of the interface plane and at all times.

From this predicate, which describes the interface in Figure 2, we can prove the geometrical properties of the wave. Firstly, the fact that the incident, reflected, and transmitted waves all lie in the same plane:

Theorem 4.5 (Law of Plane of Incidence).
$\vdash \forall \mathtt{i}\ \mathtt{emf_i}\ \mathtt{emf_r}\ \mathtt{emf_t}.\ \mathtt{is\_plane\_wave\_at\_int}\ \mathtt{i}\ \mathtt{emf_i}\ \mathtt{emf_r}\ \mathtt{emf_t}\ \wedge$
$\mathtt{non\_null}\ \mathtt{emf_r}\ \wedge\ \mathtt{non\_null}\ \mathtt{emf_t}\ \Rightarrow\ \mathtt{let}\ \mathtt{n} = \mathtt{normal\_of\_interface}\ \mathtt{i}\ \mathtt{in}$
$\mathtt{coplanar}\ \{\mathtt{vec}\ 0, \mathtt{k\_of\_w}\ \mathtt{emf_i}, \mathtt{k\_of\_w}\ \mathtt{emf_r}, \mathtt{k\_of\_w}\ \mathtt{emf_t}, \mathtt{n}\}$

A second geometric consequence is the fact that the reflected wave is symmetric to the incident wave with respect to the normal to the surface:

Theorem 4.6 (Law of Reflection).
$\vdash \forall \mathtt{i}\ \mathtt{emf_i}\ \mathtt{emf_r}\ \mathtt{emf_t}.\ \mathtt{is\_plane\_wave\_at\_int}\ \mathtt{i}\ \mathtt{emf_i}\ \mathtt{emf_r}\ \mathtt{emf_t}\ \wedge$
$\mathtt{non\_null}\ \mathtt{emf_r}\ \Rightarrow \mathtt{let}\ \mathtt{n} = \mathtt{normal\_of\_interface}\ \mathtt{i}\ \mathtt{in}\ \mathtt{are\_sym\_wrt}\ (-(\mathtt{k\_of\_w}\ \mathtt{emf_i}))\ (\mathtt{k\_of\_w}\ \mathtt{emf_r})\ \mathtt{n}$

where are_sym_wrt $\vec{u}\ \vec{v}\ \vec{w}$ formalizes the fact that $\vec{u}$ and $\vec{v}$ are symmetric with respect to $\vec{w}$ (this is easily expressed by saying that $\vec{v} = \vec{u} - 2*(\vec{u} \cdot \vec{w})\vec{w}$). Referring to Figure 2, Theorem 4.6 just means that $\theta_i = \theta_r$, which is the expression usually found in optics literatures.

We also verified the fact that the frequency is preserved by the reflected and transmitted waves:

Theorem 4.7 (Frequency Conservation).
$\vdash \forall \mathtt{i}\ \mathtt{emf_i}\ \mathtt{emf_r}\ \mathtt{emf_t}.\ \mathtt{is\_plane\_wave\_at\_int}\ \mathtt{i}\ \mathtt{emf_i}\ \mathtt{emf_r}\ \mathtt{emf_t}\ \Rightarrow$
$(\mathtt{non\_null}\ \mathtt{emf_r}\ \Rightarrow\ \omega\mathtt{\_of\_w}\ \mathtt{emf_i} = \omega\mathtt{\_of\_w}\ \mathtt{emf_r})\ \wedge\ (\mathtt{non\_null}\ \mathtt{emf_t}\ \Rightarrow\ \omega\mathtt{\_of\_w}\ \mathtt{emf_i} = \omega\mathtt{\_of\_w}\ \mathtt{emf_t})$

Next, the famous Snell's law[18] is verified as follows:

Theorem 4.8 (Snell's Law).
$\vdash \forall \mathtt{i}\ \mathtt{emf_i}\ \mathtt{emf_r}\ \mathtt{emf_t}.$
$\mathtt{is\_plane\_wave\_at\_int}\ \mathtt{i}\ \mathtt{emf_i}\ \mathtt{emf_r}\ \mathtt{emf_t}\ \wedge\ \mathtt{non\_null}\ \mathtt{emf_t}\ \Rightarrow$
$\mathtt{let}\ \mathtt{n} = \mathtt{normal\_of\_interface}\ \mathtt{i}\ \mathtt{in}$
$\mathtt{let}\ \theta = \lambda\mathtt{emf}.\ \mathtt{vectorangle}\ (\mathtt{k\_of\_w}\ \mathtt{emf})\ \mathtt{n}\ \mathtt{in}$
$\mathtt{n}_1 \sin(\theta\ \mathtt{emf_i}) = \mathtt{n}_2 \sin(\theta\ \mathtt{emf_t})$

where vectorangle $\vec{u}\ \vec{v}$ returns the angle between $\vec{u}$ and $\vec{v}$. Referring to Figure 2, Theorem 4.8 just means that $n_1 \sin(\theta_i) = n_2 \sin(\theta_r)$, which is the expression usually found in optics literatures.

Finally, another foundational result, Fresnel equations[18] in the TE mode, can be derived, which relates the magnitude of the incident and reflected waves:

Theorem 4.9 (Fresnel Equations).
$\vdash \forall \mathtt{i}\ \mathtt{emf_i}\ \mathtt{emf_r}\ \mathtt{emf_t}.$
$\mathtt{is\_plane\_wave\_at\_int}\ \mathtt{i}\ \mathtt{emf_i}\ \mathtt{emf_r}\ \mathtt{emf_t}\ \wedge\ \mathtt{non\_null}\ \mathtt{emf_r}\ \wedge\ \mathtt{non\_null}\ \mathtt{emf_t}$
$\wedge\ \mathtt{te\_mode}\ \mathtt{i}\ \mathtt{emf_i}\ \mathtt{emf_r}\ \mathtt{emf_t} \Rightarrow$
$\mathtt{let}\ (\mathtt{n1}, \mathtt{n2}, \mathtt{p}, \mathtt{n})\ =\ \mathtt{i}\ \mathtt{in}$
$\mathtt{let}\ \mathtt{mag} = \lambda\mathtt{emf}.\ (\mathsf{TE\_axis}\ \ \mathtt{i}\ \mathtt{emf_i}\ \mathtt{emf_r}\ \mathtt{emf_t} \cdot \mathsf{FST}(\mathtt{mag\_at\_pln}\ \mathtt{p}\ \mathtt{n}\ \mathtt{emf}))\ \mathtt{in}$
$\mathtt{let}\ \theta\ =\ \lambda\mathtt{emf}.\ \mathtt{vectorangle}(\mathtt{k\_of\_w}\ \mathtt{emf_i})\ \mathtt{n}\ \mathtt{in}$
$$\mathtt{mag}\ \mathtt{emf_r}\ =\ \frac{\mathtt{n1}\ \mathtt{ccos}(\theta\ \mathtt{emf_i})\ -\ \mathtt{n2}\ \mathtt{ccos}(\theta\ \mathtt{emf_t})}{\mathtt{n1}\ \mathtt{ccos}(\theta\ \mathtt{emf_i})\ +\ \mathtt{n2}\ \mathtt{ccos}(\theta\ \mathtt{emf_t})}\ \mathtt{mag}\ \mathtt{emf_i}\ \wedge$$
$$\mathtt{mag}\ \mathtt{emf_t}\ =\ \frac{\mathtt{2n1}\ \mathtt{ccos}(\theta\ \mathtt{emf_i})}{\mathtt{n1}\ \mathtt{ccos}(\theta\ \mathtt{emf_i})\ +\ \mathtt{n2}\ \mathtt{ccos}(\theta\ \mathtt{emf_t})}\ \mathtt{mag}\ \mathtt{emf_i}$$

where, `mag_at_pln` is a helper function returning the magnitude of electric and magnetic fields of a plane wave at the interface as a pair of complex numbers. The function `TE_axis` returns a unit vector with the same direction as the TE mode component of $\mathtt{emf_i}$, $\mathtt{emf_r}$, and $\mathtt{emf_t}$. The quantity $\frac{\mathtt{mag\ emf_r}}{\mathtt{mag\ emf_i}}$ is called the *reflection coefficient* of the interface `i` for the wave $\mathtt{emf_i}$. The quantity $\frac{\mathtt{mag\ emf_t}}{\mathtt{mag\ emf_i}}$ is called the *transmission coefficient* of the interface `i` for the wave $\mathtt{emf_i}$. Both these results lead to the concepts of reflectance and transmittance that are essential parameters of any optical interface. All the above results are essential to reason about any optical system. In particular, the next section shows an application whose analysis makes an essential use of Fresnel equations.

The formal proofs of the above theorems heavily rely upon complex vectors and multivariate transcendental functions properties and involved rigorous human interaction. The major advantage of these formalizations is the ability to utilize them to formally analyze optical systems, as will be demonstrated in the next section. Obviously, their development is significantly harder than their informal counterparts, especially since proofs in physics textbooks make many mathematical assumptions and simplifications that are not always justified, or are justified only by *physical* considerations without any *mathematical* arguments.

5. APPLICATION: FABRY-PÉROT RESONATOR

In this section, we show the effectiveness of our framework by formalizing the main characteristics of a Fabry-Pérot resonator,[22] shown in Figure 3. This resonator is used in many optical systems, namely, on-chip photo-detectors, lasers,[23] and optical bio-sensors.[24] This structure, fundamentally, consists of two parallel, partially reflecting mirrors with a free space between them.

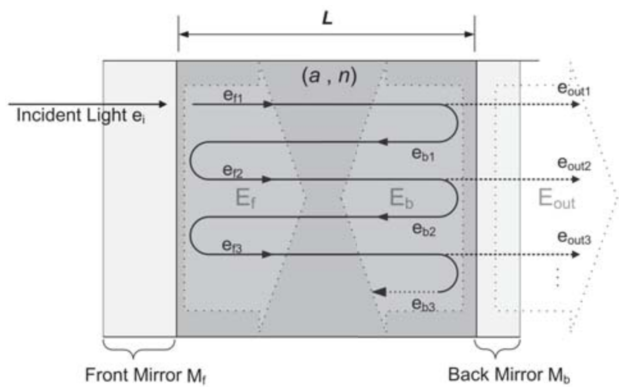

Figure 3. Fabry-Pérot Resonator

The most important concept in the Fabry-Pérot resonator is the one of constructive interference of electromagnetic fields. As it can be observed in Figure 3, the incident wave $\mathtt{E_i}$, hitting the first mirror, is partially transmitted into the free space between the mirrors. Then e_{f_1}, the transmitted electric field of the incident wave, propagates through the free space of the resonator and hits the second mirror, yielding a transmitted wave e_{out_1} and a reflected wave e_{b_1}. The same process then goes on with the wave e_{b_1}, and starts over with its reflected wave e_{f_2}. Consequently, there are infinitely many waves that are generated inside the resonator, and the waves $\mathtt{E_f}$, $\mathtt{E_b}$ and $\mathtt{E_{out}}$, as shown in Figure 3, are the (infinite) sum of all those waves.

In the next sub-section, we formally define the Fabry-Pérot resonator, then, we formally describe three electric fields, $\mathtt{E_f}$, $\mathtt{E_b}$ and $\mathtt{E_{out}}$, which are of high interest in many applications, e.g., lasers, which benefit from the properties of the light at the output of the resonator,[22] $\mathtt{E_{out}}$, and photo-detectors, which benefits from the energy of the light within the resonator,[23] which depends on $\mathtt{E_f}$ and $\mathtt{E_b}$. Finally we show how our formal model of Fabry-Pérot resonator can be practically used in more advanced structures like laser.

5.1 Formalization of the Fabry-Pérot Resonator

According to our framework flow, we need to formally describe the mediums and the EMFs involved in the system. In the Fabry-Pérot resonator, we have three mediums: one before the first mirror, one in the free space, and one after the second mirror, as shown in Figure 1. Both mirrors are modelled as interfaces between mediums. Consequently, a Fabry-Pérot resonator can be formalized by two values of type `interface`. However, to make a proper analysis of the resonator, we need to take into account the *absorption* in the free space. This means that we should also consider an *absorption coefficient* a and the width L of the space. We thus define the type `fabry_perot` as `interface` $\times$ `interface` $\times$ `real` $\times$ `real`, where the first interface is the mirror M_f, the second one is M_b, the first real number is a and the second is L. We define the predicate `is_valid_FP` to ensure that a value of type `fabry_perot` indeed represents an acceptable Fabry-Pérot resonator, by checking if `a` and `L` are positive and if both mirrors are parallel to each other and have a medium in common. All the subsequent theorems are verified under the assumption that each value of type `fabry_perot` satisfies `is_valid_FP` . This completes the formal modelling of the given medium.

The next step, according to the framework, is to formalize the EMFs. We first observe that the analysis of the resonator is meaningless where there are no reflected or transmitted waves, so in order to ensure their presence we define the following two functions which are the functional counterparts of the predicate `is_plane_wave_at_int`:

Definition 5.1 (Reflected and transmitted wave).
$\vdash_{\mathtt{def}}$ $\forall$ $\mathtt{i\ emf_i}$. $\mathtt{reflected\ i\ emf_i}$ $=$
$\quad$ @ $\mathtt{emf_r}$. $\exists$ $\mathtt{emf_t}$. $\mathtt{is\_plane\_wave\_at\_int\ \ i\ emf_i\ emf_r\ emf_t}$
$\vdash_{\mathtt{def}}$ $\forall$ $\mathtt{i\ emf_i}$. $\mathtt{transmitted\ i\ emf_i}$ $=$
$\quad$ @ $\mathtt{emf_t}$. $\mathtt{is\_plane\_wave\_at\_int\ \ i\ emf_i\ (reflected\ i\ emf_r)\ emf_t}$

where @ is the Hilbert's choice operator and given an interface `i` and a wave `emf`, `reflected i emf` and `transmitted i emf` denote the reflected and transmitted waves, respectively. Each time these functions are used, it is required that the existence of their result is asserted. For the sake of readability, we will not write such assertions in the following.

Before defining the functions returning e_{f_n} and e_{b_n} (according to the notations used in Figure 3), we need to formalize the absorption in the medium. Absorption is physically modelled as follows. Consider a wave of electric field $\vec{E}$ at point P in a medium of absorption coefficient a. Let Q be a point in the same medium, within distance d of P. Then the value of the electric field at point Q is: $e^{-\frac{ad}{2}-j\|\vec{k}\|d}\vec{E}$, where $\vec{k}$ is the wavevector. This leads to the following:

Definition 5.2 (Absorbed wave).
$\vdash_{\mathtt{def}}$ $\forall$ $\mathtt{emf\ fp\ d}$.
$\mathtt{absorbed\ emf\ fp\ d} = \lambda(\mathtt{r:point})(\mathtt{t:time}).\ \ \mathtt{e}^{-\left(\frac{\mathtt{abs\_of\_FP\ \ fp}}{2}+\mathtt{j}\|\mathtt{k\_of\_w\ \ emf}\|\right)\mathtt{d}}\mathtt{emf\ r\ t}$

where the functions `abs_of_FP` and `wdth_of_FP` retrieve the absorption coefficient and the width of the inner space, respectively. In this analysis, we are only interested in this value when the wave has traveled from one side of the resonator to the other. One can prove using some basic geometry that the corresponding travelled distance is: $\frac{L}{\cos(\vec{k},\vec{n})}$, where $\vec{n}$ is the normal to the mirrors of the Fabry-Pérot resonator. Hence, absorption in the medium is defined as:

Definition 5.3 (Absorbed in Fabry-Pérot resonator).
$\vdash_{\mathtt{def}}$ $\forall$`emf fp`. `absorbed_in_FP  emf fp` $=$
`absorbed emf (abs_of_FP fp)(trvl_dist emf (nrml_of_FP fp) (wdth_of_FP  fp))`

where `trvl_dist` is returning the distance between the two mirrors. We can now define functions returning e_{f_n} and e_{b_n} (according to the notations used in Figure 3). As already explained, these fields are the result of infinitely reflection of transmitted portion of the incident wave e_i between two mirrors and can be formalized by the following mutually recursive functions $\mathtt{e_{f_n}}$, $\mathtt{e_{b_n}}$, and $\mathtt{e_{out_n}}$:

DEFINITION 5.4 ($\mathsf{e_{f_n}}$, $\mathsf{e_{b_n}}$, AND $\mathsf{e_{out_n}}$).

$\vdash_{\mathtt{def}}\ \forall\ \mathtt{fp}\ \mathtt{emf_i}\ \mathtt{n}.\ \ \mathtt{let}\ (\mathsf{M_f},\mathsf{M_b},\mathsf{a},\mathsf{L}) = \mathtt{fp}\ \mathtt{in}$

$\mathsf{e_{f_n}}\ \mathtt{emf_i}\ \mathtt{fp}\ 0 = \mathtt{transmitted}\ \mathsf{M_f}\ \mathtt{emf_i}\ \wedge$

$\mathsf{e_{f_n}}\ \mathtt{emf_i}\ \mathtt{fp}\ (\mathtt{n}+1) =$

$\qquad\mathtt{reflected}\ \mathsf{M_f}\ (\mathtt{absorbed\_in\_FP}\ (\mathtt{reflected}\ \mathsf{M_b}\ \mathtt{absorbed\_in\_FP}\ (\mathsf{e_{f_n}}\ \mathtt{emf_i}\ \mathtt{fp}\ \mathtt{n})\mathtt{fp}))\mathtt{fp})\ \wedge$

$\mathsf{e_{b_n}}\ \mathtt{emf_i}\ \mathtt{fp}\ \mathtt{n} = \mathtt{reflected}\ \mathsf{M_b}\ (\mathtt{absorbed\_in\_FP}\ (\mathsf{e_{f_n}}\ \mathtt{emf_i}\ \mathtt{fp}\ \mathtt{n})\mathtt{fp})\ \wedge$

$\mathsf{e_{out_n}}\ \mathtt{emf_i}\ \mathtt{fp}\ \mathtt{n} = \mathtt{transmitted}\ \mathsf{M_b}\ (\mathtt{absorbed\_in\_FP}\ (\mathsf{e_{f_n}}\ \mathtt{emf_i}\ \mathtt{fp}\ \mathtt{n})\mathtt{fp})$

The waves e_{f_n}, e_{b_n} and e_{out_n} have thus been formally represented by $\mathsf{e_{f_n}}\ \mathtt{emf_i}\ \mathtt{fp}\ \mathtt{n}$, $\mathsf{e_{b_n}}\ \mathtt{emf_i}\ \mathtt{fp}\ \mathtt{n}$ and $\mathsf{e_{out_n}}\ \mathtt{emf_i}\ \mathtt{fp}\ \mathtt{n}$. Finally, the waves which we are really interested in, are actually the sums of these waves for every $n \in \mathbb{N}$:

DEFINITION 5.5 ($\mathsf{E_f}$, $\mathsf{E_b}$, AND $\mathsf{E_{out}}$).

$\vdash_{\mathtt{def}}\ \forall\ \mathtt{emf_i}\ \mathtt{fp}.\ \ \mathsf{E_f}\ \mathtt{emf_i}\ \mathtt{fp} = \lambda(\mathtt{r}:\mathtt{point})(\mathtt{t}:\mathtt{time}).\ \ \mathtt{vinfsum}\ (\mathsf{e_{f_n}}\ \mathtt{emf_i}\ \mathtt{fp}\ \mathtt{r}\ \mathtt{t})$

$\vdash_{\mathtt{def}}\ \forall\ \mathtt{emf_i}\ \mathtt{fp}.\ \ \mathsf{E_b}\ \mathtt{emf_i}\ \mathtt{fp} = \lambda(\mathtt{r}:\mathtt{point})(\mathtt{t}:\mathtt{time}).\ \ \mathtt{vinfsum}\ (\mathsf{e_{b_n}}\ \mathtt{emf_i}\ \mathtt{fp}\ \mathtt{r}\ \mathtt{t})$

$\vdash_{\mathtt{def}}\ \forall\ \mathtt{emf_i}\ \mathtt{fp}.\ \ \mathsf{E_{out}}\ \mathtt{emf_i}\ \mathtt{fp} = \lambda(\mathtt{r}:\mathtt{point})(\mathtt{t}:\mathtt{time}).\ \ \mathtt{vinfsum}\ (\mathsf{e_{out_n}}\ \mathtt{emf_i}\ \mathtt{fp}\ \mathtt{r}\ \mathtt{t})$

where $\mathtt{vinfsum}$ returns the infinite sum of a sequence of complex vectors, e.g., $\mathtt{vinfsum}\ (\mathsf{e_{out_n}}\ \mathtt{emf_i}\ \mathtt{fp})$ formalizes $\sum_{n\in\mathbb{N}} e_{out_n}$.

We have now formalized both the medium and the EMFs for the Fabry-Pérot resonator, which completes the system model. The next step is to formally specify the properties of interest.

5.2 Formal Analysis of the Fabry-Pérot Resonator

Many optical parameters of the Fabry-Pérot resonator, namely, intensity of light at the output, depend on the magnitude of the three electric fields E_f, E_b and E_{out} which we formalize as follows:

THEOREM 5.6 (MAGNITUDE OF E_f, E_b, E_{out}).

$\vdash \forall\ \mathtt{emf}\ \mathtt{fp}.\ \ \mathtt{let}\ (\mathsf{M_f},\mathsf{M_b},\mathsf{a},\mathtt{l}) = \mathtt{fp}\ \mathtt{in}$

$\mathtt{let}\ (\mathtt{emf_i},\mathtt{emf_r},\mathtt{emf_t})\ =\ (\mathtt{emf},\mathtt{reflected}\ \mathsf{M_f}\ \mathtt{emf},\mathtt{transmitted}\ \mathsf{M_f}\ \mathtt{emf})\ \mathtt{in}$

$\mathtt{is\_valid\_FP}\ \ \mathsf{M_f}\ \mathsf{M_b}\ \mathtt{a}\ \mathtt{l}\ \wedge\ \mathtt{is\_plane\_wave\_at\_int}\ \mathsf{M_f}\ \mathtt{emf_i}\ \mathtt{emf_r}\ \mathtt{emf_t}\ \wedge$

$\mathtt{tm\_mode}\ \mathsf{M_f}\ \mathtt{emf_i}\ \mathtt{emf_r}\ \mathtt{emf_t} \Rightarrow$

$\mathtt{let}\ \mathsf{n_1} = \mathtt{n1\_of\_FP}\ \mathtt{fp}\ \wedge\ \mathsf{n_2} = \mathtt{n2\_of\_FP}\ \mathtt{fp}\ \wedge\ \mathsf{n_3} = \mathtt{n3\_of\_FP}\ \mathtt{fp}\ \mathtt{in}$

$\mathtt{let}\ \mathtt{k} = \frac{\mathsf{n_1 n_2}}{\|\mathtt{k\_of\_w}\ \mathtt{emf_i}\|}\ \mathtt{in}$

$\mathtt{let}\ \theta_1 = \mathtt{vectorangle}\ (\mathtt{k\_of\_w}\ \mathtt{emf_i})\ (\mathtt{nrml\_of\_FP}\ \mathtt{fp})\ \mathtt{in}$

$\mathtt{let}\ \theta_2 = \arcsin(\frac{\mathsf{n_1}}{\mathsf{n_2}}\sin\theta_1)\ \mathtt{in}$

$\mathtt{let}\ \theta_3 = \arcsin(\frac{\mathsf{n_2}}{\mathsf{n_3}}\sin\theta_2)\ \mathtt{in}$

$\mathtt{let}\ (\mathtt{r_f},\mathtt{t_f}) = (\frac{\mathsf{n_2}\cos\theta_1 - \mathsf{n_1}\cos\theta_2}{\mathsf{n_1}\cos\theta_2 + \mathsf{n_2}\cos\theta_1}, \frac{2\mathsf{n_1}\cos\theta_1}{\mathsf{n_1}\cos\theta_2 + \mathsf{n_2}\cos\theta_1})\ \mathtt{in}$

$\mathtt{let}\ (\mathtt{r_b},\mathtt{t_b}) = (\frac{\mathsf{n_3}\cos\theta_2 - \mathsf{n_2}\cos\theta_3}{\mathsf{n_2}\cos\theta_3 + \mathsf{n_3}\cos\theta_2}, \frac{2\mathsf{n_2}\cos\theta_2}{\mathsf{n_2}\cos\theta_3 + \mathsf{n_3}\cos\theta_2})\ \mathtt{in}$

$\mathtt{let}\ \mathtt{mag} = \lambda\mathtt{emf}.\ \ (\mathtt{TM\_axis}\ \ \mathtt{i}\ \mathtt{emf_i}\ \mathtt{emf_r}\ \mathtt{emf_t} \cdot \mathsf{FST}(\mathtt{mag\_at\_pln}\ \mathtt{p}\ \mathtt{n}\ \mathtt{emf}))\ \mathtt{in}$

$\qquad\mathtt{mag}\ (\mathsf{E_f}\ \mathtt{emf_i}\ \mathtt{fp}) = \frac{\mathtt{t_f}}{1-\mathtt{r_f r_b e}^{-(\mathtt{a}+2\mathtt{jk})\mathtt{l}}}\ \mathtt{mag}\ \mathtt{emf_i}\ \wedge$

$\qquad\mathtt{mag}\ (\mathsf{E_b}\ \mathtt{emf_i}\ \mathtt{fp} = \mathtt{r_b}\mathtt{e}^{-(\frac{\mathtt{a}}{2}+\mathtt{jk})\mathtt{l}}\ \mathtt{mag}\ (\mathsf{E_f}\ \mathtt{emf_i}\ \mathtt{fp})\ \wedge$

$\qquad\mathtt{mag}\ (\mathsf{E_{out}}\ \mathtt{emf_i}\ \mathtt{fp}) = \ \frac{\mathtt{t_f t_b e}^{-(\frac{\mathtt{a}}{2}+\mathtt{jk})\mathtt{l}}}{1-\mathtt{r_f r_b e}^{-(\mathtt{a}+2\mathtt{jk})\mathtt{l}}}\ \mathtt{mag}\ \mathtt{emf_i}$

where $\mathtt{n1\_of\_FP}$, $\mathtt{n2\_of\_FP}$ and $\mathtt{n3\_of\_FP}$ return the refractive indices of the "input", free, and "output" space, respectively. The proof of this theorem involves all the primitive laws of optics of Section 4.2: the law of reflection allows us to prove that all fields e_{f_1}, e_{f_2}, ... have the same angle of incidence, and similarly for e_{b_1}, e_{b_2}, ... and e_{out_1}, e_{out_2}, ..., respectively. These angles can be retrieved using Snell's law (as can be seen in the let-bindings of θ_2 and θ_3). This entails that the coefficients of Fresnel equations for M_f and M_b are independent of e_{out_1},

$e_{out_2}, \ldots$ and similarly for $e_{b_1}, e_{b_2}, \ldots$ and $e_{out_1}, e_{out_2}, \ldots$, respectively. The resulting coefficients are bound to $\mathtt{r_f}$, $\mathtt{t_f}$, $\mathtt{r_b}$ and $\mathtt{t_b}$ in the above statement. Thus the magnitude of E_{out}, E_f and E_b can be expressed as recursive sequences depending only on $\mathtt{emf_i}$ and `FP`. Classical results on geometric series then allow us to remove the recursion and get the above expressions. Note that the use of Fresnel equations is only allowed because we fixed the polarization to TM-mode.

5.3 Use of the Fabry-Pérot Resonator in a Laser

To show the significance of our approach, in this section, we formalize the intensity of the light when the resonator is used in a laser. Generally, the optical intensity is the optical power per unit area, which is transmitted through any surface perpendicular to the propagation direction. It is proportional to the squared modulus of the electric field's norm:[17]

Definition 5.7 (Optical Intensity).
$\vdash_{\mathtt{def}}\ \forall \mathtt{emf}\ \mathtt{S}.\ \mathtt{intensity\ S\ emf}\ =\ \mathtt{S}\ \|\mathtt{e\_of\_w\ emf}\|\|\mathtt{e\_of\_w\ emf}\|^*$

where `S` is a coefficient that depends on the application.

We are now interested in evaluating the ratio between the optical intensity at the output and at the input of a Fabry-Pérot resonator. Using Theorem 5.6, we can prove the following (under the same hypotheses and using the same let-bindings):

Theorem 5.8 (Optical Intensity Ratio in a Fabry-Pérot Resonator).

$$\frac{\mathtt{intensity\ S\ (E_{out}\ emf_i\ fp)}}{\mathtt{intensity\ S\ emf_i}} = \frac{\mathtt{t_f^2 t_b^2 e^{-al}}}{(1-\mathtt{r_f r_b e^{-al}})^2(1+\frac{4\mathtt{r_f r_b e^{-al}}}{(1-\mathtt{r_f r_b e^{-al}})^2}\sin^2(\mathtt{kl}))}$$

This result allows us to know which proportion of the incident light is transmitted by the Fabry-Pérot resonator. In practice, the values of $\mathtt{t_f}$, $\mathtt{r_f}$, $\mathtt{t_b}$, $\mathtt{r_b}$ and $\mathtt{e^{-al}}$ can be directly measured. For instance, consider a Fabry-Pérot resonator with $\mathtt{t_f} = \mathtt{t_b} = \sqrt{0.1}$, $\mathtt{r_f} = \mathtt{r_b} = \sqrt{0.9}$, and $\mathtt{e^{-al}} = 0.98$. Now assuming resonator gets illuminated by a gas-laser with $\lambda_0 = 638.8nm$ and $\Delta\lambda = 40nm$, using Theorem 5.8, we can conclude that $\frac{\mathtt{intensity\ S\ (E_{out}\ emf_i\ fp)}}{\mathtt{intensity\ S\ emf_i}} > 0.7$, i.e., more than 70% of the light intensity is preserved within the considered wavelength range. This result matches the analytical approach,[22] but with the additional confidence in the result which is brought by the fact that all the steps leading to this result were checked by a theorem prover.

Overall, the corresponding HOL development is quite heavy: it required approximately one and half man-year because of the broad range of the involved areas: complex vector spaces, complex linear algebra, and, of course, electromagnetics itself. This includes the time to develop extensions to a number of standard libraries of HOL Light. More details on our developments can be found at `http://hvg.ece.concordia.ca/projects/optics/EMoptics.htm`.

6. CONCLUSION

In this paper, we presented a framework for the formal analysis of optical system based on electromagnetic models. This framework introduces the formalization of optical components using higher-order logic and the verification of desired properties in a theorem prover. The proof of these theorems is deduced from a finite set of axioms, inference rules and previously proven theorems which guarantees error-free results. Another useful aspect of the proposed framework is that it ensures the listing of all the approximations, used during the modelling phase, explicitly as the assumptions of the formally verified theorems.

Towards the development of the proposed framework, we presented a very general formalization of electromagnetic theory, and optics fundamentals like Snell's law and Fresnel equations in this paper. Moreover, we showed the practical effectiveness of our framework by formalizing the Fabry-Pérot resonator. Larger and more complicated systems could equally well be formalized, while preserving the confidence in the results.

Considering the mathematical complexity of optical model analysis, we often encounter equations with no symbolic (or "closed-form") solution. In order to handle such equations, our future work is to provide a bridge to a tool for symbolic and numerical computation such as Optica.[25] In this bridge the equation is first sent to the tool in order to be simplified symbolically. In case the equation cannot be simplified symbolically, we have no option but to switch to numerical approaches, which introduce numerical approximations to our analysis. However if the equation is simplified, the first attempt, in the theorem prover, is to certify this simplification and then to proceed with the proof. Moreover, there is priority to use symbolic computations over numerical approaches. Therefore, as far as the whole analysis is concerned, the proposed method will offer the most precise solution. The availability of this bridge will ensure that the proposed framework can effectively be employed during the development stages of the formal libraries.

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