Formal Modeling and Verification of Integrated Photonic Systems

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***Abstract*—The prominent advantages of photonics are high bandwidth, low power and the possibility of better electromagnetic interference immunity. As a result, photonics technology is increasingly used in ubiquitous applications such as telecommunication, medicine, avionics and robotics. One of the main critical requirements is to verify the corresponding functional properties of these systems. In this perspective, we identify the most widely used modeling techniques (e.g., transfer matrices, difference equations and block diagrams) for the modeling and analysis of photonic components. Considering the safety and cost critical nature of the application domain, we discuss the potential of using formal methods as a complementary analysis approach. In particular, we propose a framework to formally specify and verify the critical properties of complex photonic systems within the sound core of a higher-order-logic theorem prover. For illustration purposes, we present the formal specification of a microring resonator based photonic filter along with the verification of some important design properties such as spectral power and filtering rejection ratio.**

I. Introduction

Photonics is an extensive field of research which can be referred to the technologies that deal with the manipulation and application of light. Primary applications of photonics have emerged in remote sensing [32], biomedical imaging [6], communications [4], computing [25] and aerospace [23] to name just a few. In particular, the use of photonics in short distance communications offers the necessary improvements in communication speed and power consumption. Photonics has the potential to meet the future computing requirements by interconnecting thousands of computing nodes with Terabits/second links with an ultimate goal of Exaflops/second [25]. Recently developed silicon photonics platforms open the door for massive production of photonic devices using existing CMOS fabrication technology [20]. As a result, the cost of practical transmission systems has been significantly reduced, playing a pivotal role in the acceptance of photonics integration in diverse engineering systems.

The short time-to-market and cost associated with the fabrication process, make it impractical to analyze the influence of design parameters on the properties of the devices by successive fabrication. Moreover, the characterization of prototypes is also a time consuming process and does not unveil all of the internal behaviors of the device under test, since all properties cannot be directly measured. Therefore, it is indispensable to build detailed mathematical models which further can be used for the exhaustive analysis to understand the device operations and the dependence of device parameters. One natural step is to identify some fundamental building-blocks which are most widely used in practical photonic systems. For example, microring resonators [12] and Mach-Zehnder interferometer [27] can be considered as the core of almost all optical integrated circuits. In the optics literature, different researchers have proposed different models of the same photonic components ranging from the detailed physical equations to the more abstracted ones involving fewer parameters. Consequently, overall system models are analyzed using well-known modeling approaches such as control systems theory [17], signal processing [4] and dynamical system theory [15] as shown in Figure 1.

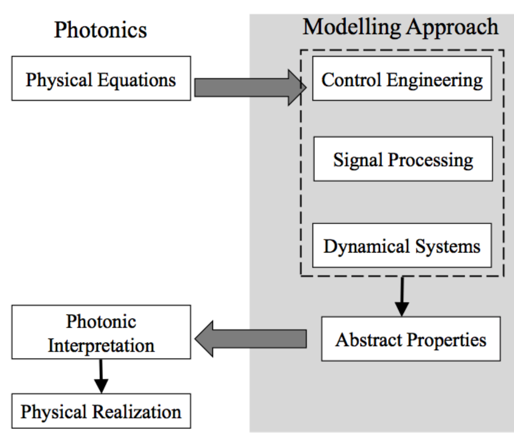

Fig. 1: Modeling Approaches for Photonic Systems

The control systems theory provides a mathematically robust set of tools for photonic system design, analysis, and control. For example, cascading photonic components requires to consider the feedback mechanism. Similarly, phtonic signal processing can be conveniently modelled using the notion of difference equations (i.e., as a feedback recurrence relation). Finally, mathematical modeling of linear photonic systems involves linear algebraic representation and can be efficiently analyzed using the dynamical systems theory. However, control theoretic and signal processing based modeling can be applied at nearly any point in the system hierarchy, whether at the

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device, subsystem, or overall network level. Each of these three modeling approaches provides the basis to analyze abstract properties of photonic systems (Figure 1). For example, control theoretic block diagram representations [17] can be used to derive overall power and crosstalk effects. The system stability and resonance can be analyzed using signal processing techniques [4]. Similarly, spectral properties such as transmissivity and reflectivity of light-waves can be described using the matrix modeling of individual photonic components [15].

Traditionally, the design of photonic systems based on the above mentioned modeling approaches has been done using the paper-and-pencil proof methods. In particular, physical equations characterizing optical systems (or components) are transformed to control theoretic block diagrams, difference equations or transfer matrices. However, the analysis of complex photonic systems using paper-and-pencil based proofs is error-prone, particularly for the case when a large number of components and interconnections are used in the optical circuits. Moreover, most of the underlying assumptions are not specified explicitly which may lead to faulty system designs. Many examples of erroneous paper-and-pencil based proofs are available in the open literature of optics (e.g., the work reported in [5] was later corrected in [19]). In recent times, high-speed computing resources are actively used to perform simulation based analysis using numerical algorithms [18], [12]. Besides the huge memory and computational time requirements, simulation cannot provide perfectly accurate results [8] due to the discritization of continuous parameters and round-off errors. Moreover, the algorithms behind these methods are not rigorous and sound. The above mentioned inaccuracy problems of traditional analysis methods are impeding their usage in designing safety-critical photonic systems, where minor bugs can lead to fatal consequences both in terms of monetary loss and risk to human life. In particular, it is more important in the applications where failures directly lead to safety issues such as in aerospace as compared to telecommunication where failures can lead to safety problems through some secondary events. An example of such a critical application is Boeing F/A-18E, for which the mission management system is linked using a photonic network [31].

Formal methods [33] are computer based reasoning techniques which allow accurate and precise analysis and thus have the potential to overcome the limitations of accuracy, found in traditional approaches. The main idea behind formal methods based analysis of systems is to develop a mathematical model for the given system and analyze this model using computer-based mathematical reasoning, which in turn increases the chances for catching subtle but critical design errors that are often ignored by traditional techniques. The two major formal methods techniques are model checking and theorem proving (a brief overview of other formal methods techniques can be found in [2]). Model checking [3] is an automated verification technique for systems that can be expressed as finite-state machines. On the other hand, higher-order logic theorem proving [9] is an interactive verification technique, but it is more flexible and can handle a variety of systems. Nowadays, the use of formal methods for high risk and safety-critical systems is recommended in many different standards like the general IEC 61508 [14] and DO178-B [24] for aviation. Ever increasing applications of photonics in safety-critical systems suggest considering the application of formal methods in this field as well. Due to the involvement of multivariate analysis and complex-valued parameters, model checking cannot be used to analyze hardware aspects of photonic systems. However, higher-order logic theorem proving can be applied in photonics due to its higher expressibility and the availability of well developed theorem provers (e.g., HOL Light [7]). In fact some preliminary work have been reported about the potential to analyze optical systems. The pioneering work about the formal analysis of optical waveguides has been reported in [10]. In [28], a preliminary infrastructure has been developed in the HOL Light theorem prover to verify some fundamental properties (e.g., ray confinement or stability) of optical systems based on ray optics. A more recent work about quantum formalization of coherent light has been reported in [16]. All of these papers describe the low level modeling of optical components which is not of great interest in the context of integrated photonic systems.

In this paper, we propose a formal framework to facilitate the formal specification and verification of widely used integrated photonic systems. Mainly we build upon our previous work [30] about the transfer matrix modeling of photonic components to widen the scope of formal methods based analysis of photonic systems. In particular, we describe the integration of generic block-diagram modeling [11] and Z-transform [29] based analysis of difference equations to analyze basic photonic components with arbitrary parameters. An exciting feature of the proposed framework is the reusability which is an important metric to evaluate industrial implications of any analysis tool. Finally, for illustration purposes, we describe the control theoretic modeling of a photonic microring resonator along with the verification of its important properties such as spectral power and rejection ratio.

The rest of the paper is organized as follows: We provide a brief overview of theorem proving and the HOL Light theorem prover in Section II. Our proposed formal analysis framework is described in Section III along with the highlights of the higher-order logic formalization of transfer matrices, difference equations and block diagrams. We describe the analysis of a photonic microring resonator as an illustrative practical application in Section IV. Finally, Section V concludes the paper.

II. Higher-order Logic-Theorem Proving

Theorem proving is concerned with the construction of mathematical theorems by a computer program using axioms and hypotheses. Theorem proving systems are widely used in software and hardware verification. For example hardware designers can prove different properties of a digital circuit by using some predicates to model the circuits. Similarly, a mathematician can prove the transitivity of real numbers using the axioms of real number theory. The language of these mathematical theorems or conjectures is logic, e.g., propositional logic, first-order logic or higher-order logic, depending upon the expressibility requirement, for example, the use of higher-order logic is advantageous over first-order logic in terms of the availability of additional quantifiers and its high expressiveness.

A theorem prover is a software in which mathematical theories can be expressed with as much accuracy as pencil-and-paper, but with the precise control of the computer which ensures that no mathematical mistake is involved. Concretely, mathematical expressions (and not just equations) can be input in the system which is able to understand its precise semantics, and thus ascertain that no incorrect reasoning is applied. As an example, a theorem prover will not allow to conclude that "$\frac{x}{x} = 1$" unless it is first proved that $x \neq 0$. No computer algebra system will take care of such a subtlety when simplifying some equation. This purely deductive aspect provides the guarantee that every sentence proved in the system is true (in particular, there is no approximation like in computer algebra systems). When a mathematical or physical theory is expressed inside a theorem prover, we say that it is formalized. There are two types of provers: automatic and interactive. In an interactive theorem prover, significant user-computer interaction is required while automatic theorem provers can perform different proof tasks automatically. The main downside of automatic theorem provers is the lack of expressiveness of the underlying logic, which limits their usage in domains where complicated mathematics is involved (e.g., multivariate calculus).

HOL Light [7] is an interactive theorem proving environment for the construction of mathematical proofs in higher-order logic. A theorem is a formalized statement that may be an axiom or could be deduced from already verified theorems by an inference rule. A theorem consists of a finite set Ω of Boolean terms called the assumptions and a Boolean term S called the conclusion. For example, "$\forall x.x \neq 0 \Rightarrow \frac{x}{x} = 1$" represents a theorem in HOL Light. A HOL Light theory consists of a set of types, constants, definitions, axioms and theorems. HOL theories are organized in a hierarchical fashion and theories can inherit the types, constants, definitions and theorems of other theories as their parents. In the development of the framework, presented in this paper, we make use of the HOL Light theories of Boolean variables, real numbers, transcendental functions and multivariate analysis. In fact, one of the primary motivations of selecting the HOL Light theorem prover for our work was to benefit from these built-in mathematical theories. The proofs in HOL Light are based on the concept of a tactic that breaks goals into simple subgoals. There are many automatic proof procedures and proof assistants available in HOL Light which help the user in verifying the decidable parts of the proof automatically.

III. Proposed Framework to Verify Photonic Systems

Our main goal is to build a framework which allows the systematic modeling and verification of basic photonic components and systems. An overview of the proposed framework is shown in Figure 2. The inputs to the proposed framework are the physical description of photonic systems and the specification of the desired properties. The first step is to represent the physical description of the given system in higher-order logic. It mainly requires the formal representation of the underlying physical (electromagnetic) equations, difference equations (recurrence relation among input and output) or block diagrams, depending on the types of photonic components. Essentially, it involves some new type definitions of photonic components and some functions that simplify the manipulation of the corresponding difference equations and block diagrams which consists of arbitrary values of the parameters (e.g., coupling constants, etc.). This step also requires the formalization of some predicates ensuring that the constructed formal model indeed represents a real system. The next step is to build the necessary machinery to conveniently analyze phtonic system models (i.e., physical equations, difference equations or block diagram) as described in the previous step. We can divide this step into three tasks: 1) Derivation of the transfer matrix (if possible) for all individual components and compose them to form the system model described in the form of field equations; 2) Formalizing the Z-transform which is a foremost requirement to analyze difference equations; 3) Formalizing the transfer function based on the block diagram representation of the photonic system. Note that this process requires theories of lists, complex numbers and multivariate analysis which are already available in HOL Light. Building upon these steps, the next requirement is to derive some generic properties of the photonic systems. For example, it is quite handy to prove theorems describing total transmissivity or reflectivity of light at input and output ports, or total power and rejection ratio of a photonic processor. Finally, we develop a library of frequently used optical components such as photonic mircoring resonators. This library greatly facilitates the formalization of new photonic systems which are composed of these components as shown in Figure 2. The output of the proposed framework is the formal proof that certifies that the system implementation meets its specification. Moreover, theorem proving based verification provides valuable insights and corner cases, in case a given property cannot be proved.

Next we present the foundational concepts of our HOL Light formalization related to the different steps of the proposed framework.

A. Physical Modeling and Formalization of Transfer Matrices

One of the primary analytical approaches to model photonic systems is to explicitly write the electromagnetic field equations for each component of the corresponding system. Then these equations are transformed into a vector-matrix model relating input and output fields. This approach is widely known as the transfer matrix modeling method [26] and offers the benefit of using complex-linear algebra for the analysis of complicated photonic systems. In the photonics literature, different fundamental components have been considered as a building block of integrated photonics (e.g., microring resonators and Mach-Zehnder interferometer [27]). However, in this paper, we consider the transfer matrix modeling of microring resonators due to their widespread usage as optical filters, optical switches, optical transistors and biosensors [12]. Photonic microresonators confine light in a closed structure by the process of total internal reflection to achieve desired functionalities such as light amplification and frequency selection. A single microring resonator can be characterized by its reflectivity (r), transmissivity (t), cavity length (L_c), power attenuation (α), wavelength λ, and effective waveguide index (n_{eff}) as shown in Figure 3.

In order to facilitate the formal reasoning process, we represent a microring resonators as a new type definition in HOL Light as follows:

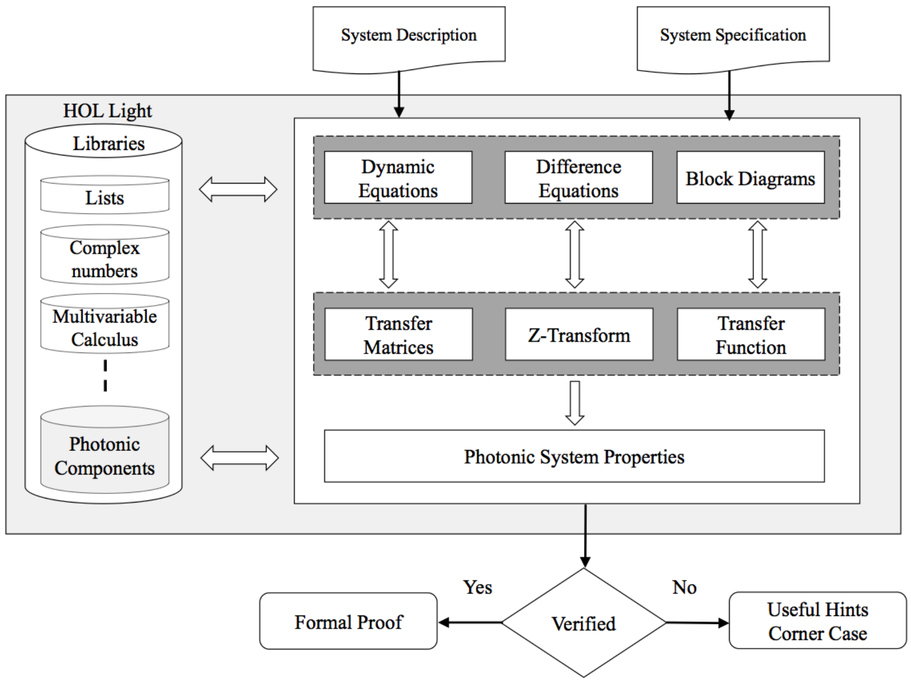

Fig. 2: Formal Framework to Verify Photonic Systems

***Definition** 1 (Microring Resonator (MRR)):*

```
new_type_abbrev "mrr =
                 ℝ × ℝ × ℝ × ℝ× ℝ × ℝ"
```

Here, the type `mrr` is a composition of six real numbers (r,t,L_c,α,λ,n_{eff}) which are necessary parameters to model a single resonator as described above.

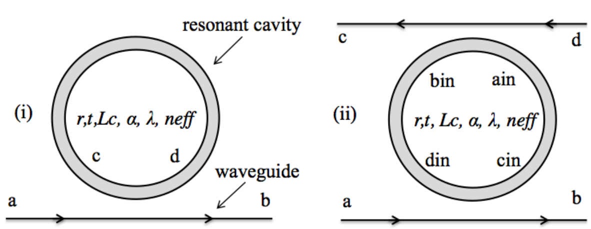

Fig. 3: Schematic Structure of Miroring Resonator

If a single ring is coupled to two bus waveguides, then the configuration represents a four-port structure as shown in Figure 3(ii), where a,b,c,d are the input, throughput, dropped and the added electromagnetic fields, respectively. We define a resonator structure by an enumerated data type in HOL Light as follows:

***Definition** 2 (Microring Resonator Structure):*

```
define_type "mrr_structure =
             two_port | four_port"
```

Next, we define what is the valid behavior of a MRR in terms of the relation between resonator parameters (r,t,L_c,α,λ,n_{eff}) and field parameters (a,b,c,d) at the input and the output. For the one-port MRR (Figure 3(i)), it is necessary to explicitly define the relation between fields inside and outside the resonator. On the other hand, it is sufficient to model the physical behavior using the two input and two output fields in case of a four-port MRR (Figure 3(ii)) [15]. Then the predicate is defined by case analysis on the MRR structure:

***Definition** 3 (Valid Behavior in MRR Structures):*

$$\vdash (\texttt{is\_valid\_behavior\_in\_mrr}\ (a,d)\ (b,c)\ (r,t,L_c,\alpha,\lambda,n_{eff}):\texttt{mrr four\_port} \Leftrightarrow$$
$$\texttt{let}\ \delta = (\tfrac{2*\pi}{\lambda}) * n_{eff} * L_c\ \texttt{and}\ \tau = \exp(-\tfrac{\alpha*Lc}{2})\ \texttt{in}$$
$$\texttt{let}\ R = -\frac{r*(1-\tau*\exp(-j*\delta))}{1-r^2*\tau*\exp(-j*\delta)}\ \texttt{and}$$
$$T = -\frac{t^2*\sqrt{(\tau)}*\exp(-\frac{j*\delta}{2})}{1-r^2*\tau*\exp(-j*\delta)}\ \texttt{in}$$
$$d = \tfrac{1}{R} * c - \tfrac{T}{R} * a \wedge b = \tfrac{T}{R} * c + \tfrac{R^2-T^2}{R} * a) \wedge$$
$$(\texttt{is\_valid\_behavior\_in\_mrr}\ (a,d)\ (b,c)\ (r,t,L_c,\alpha,\lambda,n_{eff}):\texttt{mrr two\_port} \Leftrightarrow$$
$$c = -\tfrac{1}{j*t} * (a + r * b) \wedge d = \tfrac{1}{j*t} * (r * a + b))$$

Here, `is_valid_behavior_in_mrr` takes the four fields parameters ($a,b,c,d \in \mathbb{C}$), a microring resonator ($r,t,L_c,\alpha,\lambda,n_{eff}$) and `mrr_structure`, and returns the relation among these parameters. Note that j represents an imaginary unit and $j^2 = -1$. The parameter δ represents the frequency-dependent phase shift, τ represents the waveguide loss effect, T and R represent the output field in the backward direction and forward direction, respectively.

The transfer matrix modeling [26] is the most widely used approach to analytically model MRRs [2]. The main characteristics of this technique are to decompose photonic circuits in the form of series of MRRs and then analyzing different behaviors using complex matrix algebra. Now, equipped with the above formal definitions (Definitions 1-3), we verify the transfer matrix relation of MRRs in case of two-port and four-port structures [15].

***Theorem** 1 (MRR Matrix for Two-Port Structure):*
⊢ ∀a b c d **r t** L_c α λ n_{eff}.
  is_valid_behavior_in_mrr (a,d) (b,c)
  $(\mathtt{r},\mathtt{t},\mathtt{L_c},\alpha,\lambda,\mathtt{n_{eff}})$:mrr two_port ⟹

$$\begin{bmatrix} c \\ d \end{bmatrix} = \frac{1}{j*t}\begin{bmatrix} -1 & -r \\ r & 1 \end{bmatrix} ** \begin{bmatrix} a \\ b \end{bmatrix}$$

On the similar lines, we have also derived the matrix model of MRR in case of four port structure. Morover, a comprehensive formalization to compose multiple resonators in a 2-D structure, for the verification of transmissivity and refelectivity of light radiations, have also been developed and more details can be found in [30].

B. Difference Equations and Formalization of Z-Transform

The most fundamental concept behind the signal processing capabilities of photonic components is multiple and temporal interference of delayed optical signals. By controlling the amplitudes and the phase of the interfered signals, the information in the form of optical signal can be processed. A schematic representation of an arbitrary photonics circuit is shown in Figure 4. The input optical signal is launched on the left side and split into multiple waveguides. The individual optical signal experiences equally different time delays represented by Z^{-1}. The amplitude and phase changes are represented by coefficients α_i and β_i. These signals are combined to generate the output signal at the output port. A difference equation characterizes the behavior of a particular phenomena over a period of time. Such equations are widely used to mathematically model complex dynamics of discrete-time systems. Indeed, a difference equation can capture the dynamics of the circuit representation described in Figure 4, as follows:

$$y[n] = \sum_{i=1}^{M} \alpha_i y[n-i] + \sum_{i=0}^{N} \beta_i x[n-i] \tag{1}$$

where α_i and β_i are input and output coefficients. The output $y[n]$ is a linear combination of the previous M output samples, the present input $x[n]$ and N previous input samples. In case of a time-invariant filter, α_i and β_i are considered constants (either complex ($\mathbb{C}$) or real ($\mathbb{R}$)) to obtain the output response according to the given specifications. Z-transform [22] provides a mechanism to map discrete-time signals over the complex plane, i.e., the z-domain. This transform is a powerful tool to solve linear difference equations by transforming them into algebraic operations in z-domain.

Mathematically, Z-transform can be defined as a function series as follows:

$$X(z) = \sum_{n=0}^{\infty} f[n] z^{-n} \tag{2}$$

where $f[n]$ is a complex-valued function ($f : \mathbb{N} \rightarrow \mathbb{R}$) and the series is defined for those $z \in \mathbb{C}$ for which the series is convergent. The application of Z-transform properties of time delay and advance results in the following transformation of difference representation of input as follows:

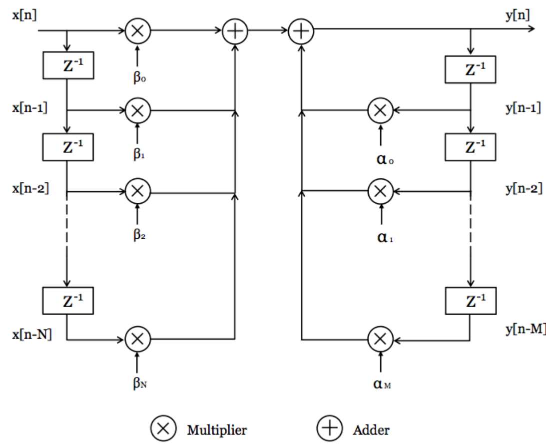

Fig. 4: Schematic Representation of Photonic Signal Processing Component

$$\mathcal{Z}(\sum_{i=0}^{M} \alpha_i x[n-i]) \; z = X(z) \sum_{i=0}^{M} \alpha_i z^{-n} \tag{3}$$

We formalize the difference equation as follows:

***Definition** 4 (Difference Equation):*
⊢ ∀ N α_lst f x.
  difference_eq M α_lst f x =
  $\sum_{t=0}^{M}$ (λt. EL t α_lst * f (x - t))

The function `difference_eq` accepts the order (`M`) of the difference equation, a list of coefficients α_`lst`, a causal function `f` and the variable `x`. It utilizes the function `EL i L`, which returns the i^{th} element of a list `L`, to generate the difference equation corresponding to the given parameters. Now, we can formalize the Z-transform function (Equation 2) in HOL Light, as follows:

***Definition** 5 (Z-Transform):*
⊢ ∀ f z. **z_transform** f z =
        infsum (from 0) (λ n. f n * z^{-n})

where the `z_transform` function accepts two parameters: a function $\mathtt{f} : \mathbb{N} \rightarrow \mathbb{C}$ and a complex variable $\mathtt{z} : \mathbb{C}$. It returns a complex number which represents the Z-transform of f according to Equation (2).

We verified a number of properties of Z-transform such as linearity, time scaling, differentiation, time advance and time delay which can be used to transform difference equations into z-domain and help to obtain the analytical solution. For the sake of conciseness, we do not provide these verification details in this paper and more detailed presentation and discussion about these properties can be found in [29]. However, we present an important property which describes the transformation of difference equation into its corresponding z-domain representation, as follows:

Theorem *2 (Z-Transform of Difference Equation):*

$$\vdash \forall\ \texttt{M}\ \alpha\texttt{\_lst}\ \texttt{f x. z} \in \texttt{ROC f} \wedge \texttt{z} \neq \texttt{Cx(\&0)} \wedge$$
$$\texttt{(is\_causal f m)} \Longrightarrow$$
$$\texttt{z\_transform} (\lambda\texttt{x.}$$
$$\texttt{difference\_eq M}\ \alpha\texttt{\_lst f x) z} =$$
$$\texttt{(z\_transform f z)} \ast$$
$$(\textstyle\sum_{\texttt{i=0}}^{\texttt{M}} (\lambda\ \texttt{n. EL n}\ \alpha\texttt{\_lst} \ast \texttt{z}^{-\texttt{n}}))$$

where ROC represents the region of convergence which provides the set of all the values for which z-transform converges.

C. Block Diagrams and Formalization of Transfer Functions

Dynamics of many engineering systems can be represented by the graphical representation of interconnected modules of those systems. Block diagram [21] modeling is a graphical method to describe the overall behaviour of a control system. All subsystems (or components) are represented as blocks, representing the relation among the input and output signals. These blocks can be arranged in various ways depending on the physical description of the underlying system. When multiple subsystems are connected such that the output of one subsystem serves as the input to the next, these subsystems are said to be in cascade form as shown in Figure 5 (a). On the other hand, parallel subsystems have a common input and their outputs are summed together (Figure 5 (b)). The feedback mechanism can be represented using feeding the output signals back to the input of the system (Figure 5 (c)). Inspired by the generic nature of the block diagram method, many researchers have used them to model practical photonic systems, e,g., photonic filters [4]. The main idea is to model the complicated subsystems of photonic systems as blocks which further can be reduced using the simplification rules as described in Figure 5. Generally, the reduced model is called the transfer function which represents the mathematical relation among the output and input of the system. Based on the transfer function, many critical properties of the system can be verified such as stability (bounded-input and bounded-output), power spectrum and delay dynamics of the system.

We next provide some formal definitions corresponding to the basic building blocks of control systems given in Figure 5. The availability of these definitions provide the foundations to formalize the block diagrams of any control system. The transfer function of two subsystems connected in series (or cascaded configuration) can be formalized as:

Definition *6 (Series Connection):*

$$\vdash \forall\ \texttt{X}_i\texttt{.}\ \textbf{series}\ [\texttt{X}_1\texttt{;}\ \texttt{X}_2\texttt{;}\ldots\texttt{;}\ \texttt{X}_N] = \prod_{i=1}^{N} \texttt{X}_i$$

where the function series accepts a list of complex numbers, which represent the transfer functions of individual subsystems, and recursively returns their product.

The summation junction is an addition module that adds the transfer functions of all the incoming branches.

Definition *7 (Summing Junction):*

$$\vdash \forall\ \texttt{X}_i\texttt{.}\ \textbf{sum\_j}\ [\texttt{X}_1\texttt{;}\ \texttt{X}_2\texttt{;}\ldots\texttt{;}\ \texttt{X}_N] = \sum_{i=1}^{N} \texttt{X}_i$$

Next, we define the feedback function as follows:

Definition *8 (Feedback):*

$$\vdash \forall\ \texttt{X Y.}\ \textbf{feedback}\ \texttt{X Y} =$$
$$\texttt{series [X;}\ \textstyle\sum_{k=0}^{\infty} \texttt{branch X Y k]}$$

where the function feedback accepts the forward path transfer function X and the feedback path transfer function Y and returns the net transfer function by forming the series network of the summation of all the possible infinite branches and the final forward path transfer function.

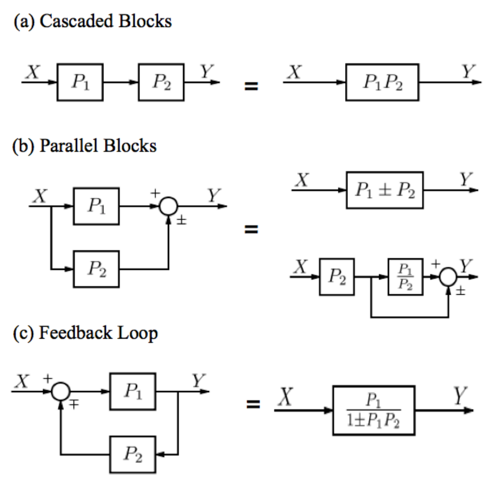

Fig. 5: Some Block Diagram Simplification Rules

In order to provide the reasoning support for the above mentioned definitions, and thus minimize human interaction during the formal verification of the overall transfer function, we prove the following theorems.

Theorem *3 (Feedback Simplification):*

$$\vdash \forall\ \texttt{X Y.}\ \|\texttt{X} \ast \texttt{Y}\| < 1 \Longrightarrow$$
$$\texttt{feedback X Y} = \frac{\texttt{X}}{1 - \texttt{X} \ast \texttt{Y}}$$

where $\|\ .\ \|$ represents the complex norm which is available in the multivariate analysis libraries of HOL Light.

Theorem *4 (Feedback Loop Simplification Rule):*

$$\vdash \forall\ \texttt{X Y Z.}\ \|\texttt{X} \ast \texttt{Y}\| + \|\texttt{X} \ast \texttt{Z}\| < 1 \wedge$$
$$\|\texttt{X} \ast \texttt{Y}\| < 1 \Longrightarrow$$
$$\texttt{feedback X (sum\_j (pickoff 1 [Y;Z]))} =$$
$$\texttt{feedback (feedback X Y) Z}$$

where the function pickoff represents a submodule connected to a parallel branch of the submodules. Similarly, more helping theorems about the simplification of block diagrams and equivalences among different structure have been verified, where more details can be found in [1].

IV. Application: Spectral Response of a Photonic Microring Resonator

The spectral response describes the sensitivity of the photonic components to the optical radiation of different wavelengths. Generally, the spectral response of photonic components can be analyzed by constructing an equivalent control theoretic model of the photonic components. The main task is to compute the ratio of the output and input field intensities, which is equivalent to the transfer function of block diagram representation of that photonic system. We consider the four port configuration of the MRR which consists of one ring and two waveguides linked using the directional couplers [13]. The description of the couplers and waveguides that make up the microresonator can be combined into a full model in the form of a block diagram as shown in Figure 6.

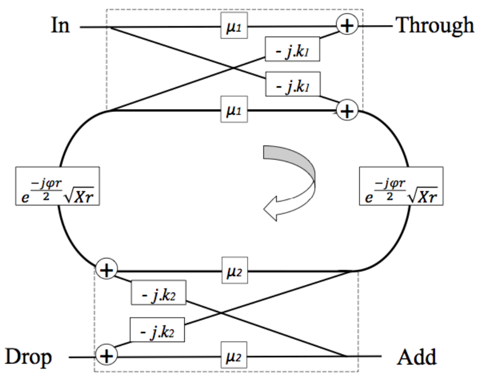

Fig. 6: Control Theoretic Model of an MRR

The four ports of the resonator are named as *input*, *through*, *add* and *drop*. The spectral response of this resonator model can be divided into two parts: 1) The drop response, which signifies the power fraction of the light that is transferred by the resonator from the In port to the Drop port as a function of the wavelength of this light; 2) The through response, which signifies the power fraction of the light that is extracted by the resonator from the In port to the Through port. The explicit path description of the drop and through response is shown in Figures 7(a) and 7(b), respectively. Next, we only consider the drop response and present the formal modeling and the verification of its power and rejection ratio.

Definition *9 (MRR Drop Response):*

```
⊢ ∀ phi xr k1 k2 u1 u2.
  mrr_drop_response phi xr k1 k2 u1 u2 =
  series[-j*k1;
  feedback ((exp(-j*phi/2))*sqrt (xr))
  (series [u1;
  exp(-j* phi/2)*(sqrt (xr)); u2]);
  -j*k2]
```

where the function `mrr_drop_response` takes six parameters of the MRR and returns a complex value describing the block diagram representation as shown in Figure 7(a).

The power of the drop response can be defined as the magnitude (square of the complex norm) of the transfer function as follows:

Definition *10 (MRR Drop Response Power):*

```
⊢ ∀ phi xr k1 k2 u1 u2.
  drop_response_power phi xr k1 k2 u1 u2 =
  ‖mrr_drop_response phi xr k1 k2 u1 u2‖^2
```

One of the most important parameters in photonic filter design is the rejection ratio which quantifies the ability to reject unwanted signals. The filter rejection ratio of a microring resonator can be defined as the ratio of the power that is transferred on resonance (phi = 2π) and the power that is transferred by the resonator when it is completely off resonance (phi = π). We formally encode this description in HOL Light as follows:

Definition *11 (Rejection Ratio):*

```
⊢ ∀ xr k1 k2 u1 u2.
  rejection_ratio xr k1 k2 u1 u2 =
  10*log *
  drop_response_power 2*π xr k1 k2 u1 u2
  ---------------------------------------
   drop_response_power π xr k1 k2 u1 u2
```

where rejection ratio is defined in decibel (DB) units which is a logarithmic scale used to express the ratios of two physical quantities. Our next task is to formally derive generic

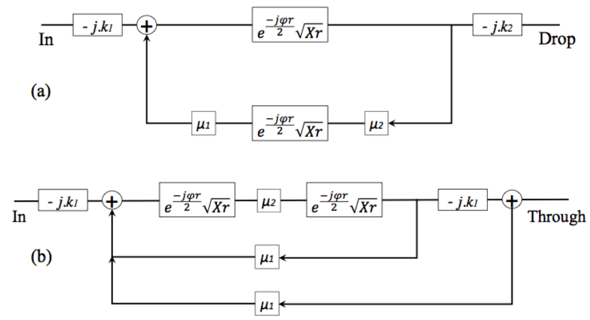

Fig. 7: The Drop Response of MRR

expressions for the above mentioned properties. As a first step, we verify the equivalent transfer function of the drop response as follows:

Theorem *5 (Transfer Function):*

```
⊢ ∀ phi xr k1 k2 u1 u2.
  0 < xr ∧ ‖xr * u1 * u2‖ < 1 ⟹
  mrr_drop_response phi xr k1 k2 u1 u2 =
  -(k1*k2)*exp(-j* phi/ 2) * sqrt (xr)
  ------------------------------------
       1 - u1*u2*exp(-j* phi) * xr
```

where both assumptions are required to evaluate the feedback loops. We prove the above result using Theorems 3 and 4. We next utilize Theorem 5 to verify the analytical result for the MRR power and rejection ratio.

Theorem *6 (Drop Response Power Verification):*

```
⊢ ∀ phi xr k1 k2 u1 u2.
  0 < xr ∧ ‖xr * u1 * u2‖ < 1 ⟹
  drop_response_power phi xr k1 k2 u1 u2 =

                      (k1*k2)^2*xr
---------------------------------------------------------
(1-u1*u2*xr)^2+4*k1*k2*cexp(-phi)*sin^2(phi/2)
```

Theorem *7 (Rejection Ratio):*

```
⊢ ∀ xr k1 k2 u1 u2.
  0 < xr ∧ ‖xr * u1 * u2‖ < 1 ⟹
  rejection_ratio xr k1 k2 u1 u2 =
          (1 + u1*u2*xr)^2
  10*log*-----------------
          (1 - u1*u2*xr)^2
```

This completes our formal verification of the spectral response of an MRR which is widely used for the industrial integrated photonics applications. The power and rejection ratio expressions have been verified under the general parameters of the MRR structure (e.g., u1, u2, etc.) which is not possible in the case of simulation [13], where these properties can only be verified for particular values. Note that the analysis presented in this paper is only for one MRR, however, our formalization is generic and can be applied for cascaded applications where N resonators are arranged in series, in parallel or in a mixed configuration [13]. Another benefit of the theorem proving based approach is to unveil all the hidden assumptions required to verify the desired property of the system which are often ignored by paper-and-pencil based proofs. For example, both assumptions mentioned in Theorems 5, 6 and 7 are not mentioned in [13]. We have identified similar critical missing assumptions during the formal verification of 2-D MRR based filters reported in [30]. Considering the safety-critical applications of photonic systems, such an analysis can be very dangerous and thus may lead to fatal consequences. On the other hand, theorem proving can be used to validate paper-and-pencil based proofs and certify the results obtained by simulation. However, theorem proving is not popular in photonics industry due to several factors such as unfamiliarity of formal methods by physicists and the high cost of using formal methods. We believe that the work reported in this paper can be considered as a one step towards the long term goal of reducing the gap among photonics and formal methods communities. Moreover, continuous work in this direction can result in more formalized libraries and verified photonic components which can ultimately ease the process of adapting theorem proving in the fastest growing high-tech photonics industries worldwide.

V. Conclusion

In this paper, we discussed the emergence of photonics and its impact on industrial applications. In particular, we provided a brief overview of different modeling approaches (e.g., transfer matrices, signal processing and control theory) used for various integrated photonic components and systems. We also described the fundamental formalizations of transfer matrices, difference equations and block diagrams and how they can be applied to photonics. Lastly, we demonstrated the potential use of our proposed framework by the formal verification of some properties (i.e., spectral power and rejection ratio) of a widely used photonic microring resonator.

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