Formal Analysis of Optical Waveguides in HOL

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Abstract. Optical systems are becoming increasingly important as they tend to resolve many bottlenecks in the present age communications and electronics. Some common examples include their usage to meet high capacity link demands in communication systems and to overcome the performance limitations of metal interconnect in silicon chips. Though, the inability to efficiently analyze optical systems using traditional analysis approaches, due to the continuous nature of optics, somewhat limits their application, specially in safety-critical applications. In order to overcome this limitation, we propose to formally analyze optical systems using a higher-order-logic theorem prover (HOL). As a first step in this endeavor, we formally analyze eigenvalues for planar optical waveguides, which are some of the most fundamental components in optical devices. For the formalization, we have utilized the mathematical concepts of differentiation of piecewise functions and one-sided limits of functions. In order to illustrate the practical effectiveness of our results, we present the formal analysis of a planar asymmetric waveguide.

1 Introduction

Optical systems are increasingly being used these days, mainly because of their ability to provide high capacity communication links, in applications ranging from ubiquitous internet and mobile communications, to not so commonly used but more advanced scientific domains, such as optical integrated circuits, biophotonics and laser material processing. The correctness of operation for these optical systems is usually very important due to the financial or safety critical nature of their applications. Therefore, quite a significant portion of the design time of an optical system is spent on analyzing the designs so that functional errors can be caught prior to the production of the actual devices. Calculus plays a significant role in such analysis. Nonliner differential equations with transcendental components are used to model the electric and magnetic field components of the electromagnetic light waves. The optical components are characterized by their refractive indices and then the effects of passing electromagnetic waves of visible and infrared frequencies through these mediums are analyzed to ensure that the desired reflection and refraction patterns are obtained.

The analysis of optical systems has so far been mainly conducted by using paper-and-pencil based proof methods [18]. Such traditional techniques are usually very tedious and always have some risk of an erroneous analysis due to the

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complex nature of the present age optical systems coupled with the human-error factor. The advent of fast and inexpensive computational power in the last two decades opened up avenues for using computers in the domain of optical system analysis. Nowadays, computer based simulation approaches and computer algebra systems are quite frequently used to validate the optical system analysis results obtained earlier via paper-and-pencil proof methods. In computer simulation, complex electromagnetic wave models can be constructed and then their behaviors in an optical medium of known refractive index can be analyzed. But, computer simulation cannot provide 100% precise results since the fundamental idea in this approach is to approximately answer a query by analyzing a large number of samples. Similarly, computer algebra systems, which even though are considered to be semi-formal and are very efficient in mathematical computations, also fail to guarantee correctness of results because they are constructed using extremely complicated algorithms, which are quite likely to contain bugs. Thus, these traditional techniques should not be relied upon for the analysis of optical systems, especially when they are used in safety critical areas, such as medicine, transportation and military, where inaccuracies in the analysis may even result in the loss of human lives.

In the past couple of decades, formal methods have been successfully used for the precise analysis of a verity of hardware and software systems. The rigorous exercise of developing a mathematical model for the given system and analyzing this model using mathematical reasoning usually increases the chances for catching subtle but critical design errors that are often ignored by traditional techniques like simulation. Given the sophistication of the present age optical systems and their extensive usage in safety critical applications, there is a dire need of using formal methods in this domain. However, due to the continuous nature of the analysis and the involvement of transcendental functions, automatic state-based approaches, like model checking, cannot be used in this domain. On the other hand, we believe that higher-order-logic theorem proving offers a promising solution for conducting formal analysis of optical systems. The main reason being the highly expressiveness nature of higher-order logic, which can be leveraged upon to essentially model any system that can be expressed in a closed mathematical form. In fact, most of the classical mathematical theories behind elementary calculus, such as differentiation, limit, etc., and transcendental functions, which are the most fundamental tools for analyzing optical systems, have been formalized in higher-order logic [6]. Though, to the best of our knowledge, formal analysis of optical devices is a novelty that has not been presented in the open literature so far using any technique, including theorem proving.

In this paper, as a first step towards using a higher-order-logic theorem prover for analyzing optical systems, we present the formal analysis of planar optical waveguides operating in the *transverse electric* (TE) mode, i.e., a mode when electric field is transverse to the plane of incidence. A waveguide can be defined as an optical structure that allows the confinement of electromagnetic light waves within its boundaries by *total internal reflection* (TIR). It is considered to be one of the most fundamental components of any optical system. Some of the optical systems that heavily rely on optical waveguides, include fiber-optic communications links, fiber lasers and amplifiers for high-power applications, as well as all optical integrated circuits. A planar waveguide, which we mainly analyze in this paper, is a relatively simple but widely used structure for light confinement. It is well accepted in the optics literature that the one-dimensional analysis of this simple planar waveguide is directly applicable to many real problems and the whole concept forms a foundation for more complex optical structures [18].

In order to formally describe the behavior of the planar waveguide, we model the electric and magnetic field equations, which govern the passage of light waves through a planar waveguide, in higher-order logic. The formalization is relatively simple because in the TE mode there is no $y - axis$ dependance, which allows us to describe the electromagnetic fields as a small subset of Maxwell Equations. Based on these formal definitions, we present the verification of the eignevalue equation for a planar waveguide in the TE mode. This equation plays a vital role in designing planar waveguides, as it provides the relationship between the wavelength of light waves that need to be transmitted through a planar waveguide and the planar waveguide's physical parameters, such as refractive indices and dimensions. In this formalization and verification, we required the mathematical concepts of differentiation of piecewise functions and one-sided limits. We built upon Harrison's real analysis theories [6] for this purpose, which include the higher-order-logic formalization of differentiation and limits. We also present some new definitions that allow us to reason about the differentiation of piecewise functions and one-sided limits with minimal reasoning efforts. Finally, in order to illustrate the effectiveness of the formally verified eigenvalue equation in designing real-world optical systems, we present the analysis of a planar dielectric structure [18]. All the work described in this paper is done using the HOL theorem prover [4]. The main motivations behind this choice include the past familiarity with HOL along with the availability of Harrison's real analysis theories [6], which forms the fundamental core of our work.

The remainder of this paper is organized as follows. Section 2 gives a review of related work. In Section 3, we provide a brief introduction about planar waveguides along with their corresponding electromagnetic field equations and eigenvalues. In Section 4, we present the formalization of the electromagnetic fields for a planar waveguide. We utilize this formalization to verify the eigenvalue equation in Section 5. The analysis of a planar dielectric structure is presented in Section 6. Finally, Section 7 concludes the paper.

2 Related Work

The continuous advancement of optical devices towards increased functionality and performance comes with the challenge of developing analysis tools that are able to keep up with the growing level of sophistication. Even though, there is a significant amount of research going on in this important area of analyzing optical systems but, to the best of our knowledge, none of the available optical analysis tools are based on formal methods and the work presented in this paper is the first one of its kind. In this section, we present a brief overview of the state-of-the-art informal techniques used for optical system analysis.

The most commonly used computer based techniques for optical system analysis are based on simulation and numerical methods. Some examples include the analysis of integrated optical devices [20], optical switches [16] and biosensors [23]. Optical systems are continuous systems and thus the first step in their simulation based analysis is to construct a discrete model of the given system [5]. Once the system is discretized, the electromagnetic wave equations are solved by numerical methods. Finite difference methods are the most commonly used numerical approaches applied on wave equations. Finite difference methods applied to the time domain discretized wave equations are referred to as the Finite Difference Time Domain (FDTD) methods [21] and to the frequency domain discretized wave equations as the Finite Difference Frequency Domain (FDFD) methods [19]. Solving equations with numerical methods itself imposes an additional form of error on solutions of the problem. Besides inaccuracies, another major disadvantage, associated with the numerical methods and simulation based approaches, is the tremendous amount of CPU time and memory requirements for attaining reasonable analysis results [10]. In [9,13], the authors argued different methodologies to break the structure into smaller components to improve the memory consumption and speed of the FDTD methods. Similarly, some enhancements for the FDFD method are proposed in [22,12]. There is extensive effort on this subject and although there are some improvements but the inherent nature of numerical and simulation based methods fails all these effort to bring 100% accuracy in the analysis, which can be achieved by the proposed higher-order-logic theorem proving based approach.

Computer algebra systems incorporate a wide variety of symbolic techniques for the manipulation of calculus problems. Based on these capabilities, they have been also tried in the area of optical system analysis. For example, the analysis of planar waveguides using Mathematica [14], which is a widely used computer algebra system, is presented in [3]. With the growing interest in optical system analysis, a dedicated optical analysis package *Optica* [17] has been very recently released for Mathematica. Optica performs symbolic modeling of optical systems, diffraction, interference, and Gaussian beam propagation calculations and is general enough to handle many complex optical systems in a semi-formal manner. Computer algebra systems have also been found to be very useful for evaluating eigenvalues for transcendental equations. This feature has been extensively used along with the paper-and-pencil based analytical approaches. The idea here is to verify the eigenvalue equation by hand and then feed that equation to a computer algebra system to get the desired eigenvalues [18]. Despite all these advantages, the analysis results from computer algebra systems cannot be termed as 100% precise due to the many approximations and heuristics used for automation and reducing memory constraints. Another source of inaccuracy is the presence of unverified huge symbolic manipulation algorithms in their core, which are quite likely to contain bugs. The proposed theorem proving based approach overcomes these limitations but at the cost of significant user interaction.

3 Planar Waveguides

Planar waveguides are basically optical structures in which optical radiation propagates in a single dimension. Planar waveguides have become the key elements in the modern high speed optical networks and have been shown to provide a very promising solution to overcome performance limitations of metal interconnect in silicon chips.

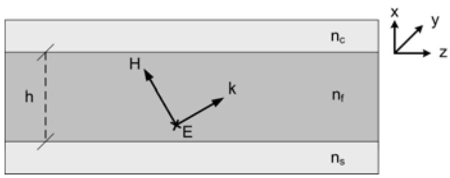

Fig. 1. Planar Waveguide Structure

The planar waveguide, shown in Figure 1, is considered to be infinite in extent in two dimensions, lets say the yz plane, but finite in the x direction. It consists of a thin dielectric film surrounded by materials of different refractive indices. The refractive index of a medium is usually defined as the ratio between the phase velocity of the light wave in a reference medium to the phase velocity in the medium itself and is a widely used characteristic for optical devices. In Figure 1, n_c, n_s, and n_f represent the refractive indices of the cover region, the substrate region, and the film, which is assumed to be of thickness h, respectively. The refractive index profile of a planar waveguide can be summarized as follows:

$$n(x) = \begin{cases} n_c & x > 0 \\ n_f & -h < x < 0 \\ n_s & x < -h \end{cases} \tag{1}$$

The most important concept in optical waveguides is that of *total internal reflection* (TIR). When a wave crosses a boundary between materials with different refractive indices, it is usually partially refracted at the boundary surface, and partially reflected. TIR happens when there is no refraction. Since, the objective of waveguides is to guide waves with minimum loss, ideally we want to ensure TIR for the waves that we want the waveguide to guide. TIR is ensured only when the following two conditions are satisfied. Firstly, the refractive index of the transmitting medium must be greater than its surroundings, $n_{medium} > n_{surrounding}$ and secondly, the angle of incidence of the wave at the medium is greater than a particular angle, which is usually referred to as the *critical angle*. The value of the critical angle also depends on the relative refractive index of the two materials of the boundary. Thus, the distribution of refractive indices of the waveguides characterize the behavior of the waveguide and restricts the type of waves which the waveguide can guide.

Like all other waveguides, the planar waveguide also needs to provide the TIR conditions for the waves, which are required to be transmitted through them. The first condition is satisfied by choosing n_f to be greater than both n_s and n_c. The second condition, on the other hand, is dependent on the angle of incidence of the wave on the boundary of the waveguide and thus involves the characteristics of the wave itself, which makes it more challenging to ensure.

Basically, light is an electromagnetic disturbance propagated through the field according to electromagnetic laws. Thus, propagation of light waves through a medium can be characterized by their electromagnetic fields. Based on Maxwell equations [11], which completely describe the behavior of light waves, it is not necessary to solve electromagnetic problems for each and every field component. It is well known that for a planar waveguide, it suffices to consider two possible electric field polarizations, *transverse electric* (TE) or *transverse magnetic* (TM) [18]. In the TE mode, the electric field is transverse to the direction of propagation and it has no longitudinal component along the *z-axis*. Thus, the $y - axis$ component of the electric field E_y is sufficient to completely characterize the planar waveguide. Similarly, in the TM mode, magnetic field has no longitudinal components along the *z-axis* and solving the system only for the $y - axis$ component of the magnetic field H_y will provide us with the remaining electric field components. In this paper, we focus on the TE mode, though the TM mode can also be analyzed in a similar way.

Based on the above discussion, the electric and magnetic field amplitudes in the TE mode for the three regions, with different refractive indices, of the planar waveguide are given as follows [18]:

$$E_y(x) = \begin{cases} Ae^{\gamma_c x} & x > 0 \\ B\cos(\kappa_f) + C\sin(\kappa_f x) & -h < x < 0 \\ De^{\gamma_s(x+h)} & x < -h \end{cases} \tag{2}$$

$$H_z = \frac{j}{\omega\mu_0}\frac{\partial E_y}{\partial x} \tag{3}$$

where A, B, C, and D are amplitude coefficients, γ_c and γ_s are *attenuation coefficients* of the cover and substrate, respectively, κ_f is the *transverse component* of the wavevector $k = \frac{2\pi}{\lambda}$ in the guiding film, ω is the angular frequency of light and μ is the permeability of the medium. Some of these parameters can be further defined as follows:

$$\gamma_c = \sqrt{\beta^2 - k_0^2 n_c^2} \tag{4}$$

$$\gamma_s = \sqrt{\beta^2 - k_0^2 n_s^2} \tag{5}$$

$$\kappa_f = \sqrt{k_0^2 n_f^2 - \beta^2} \tag{6}$$

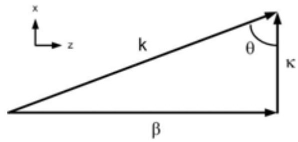

Fig. 2. Longitudinal (β) and Transverse (κ) Components of Wavevector k

where k_0 is the vacuum wavevector, such that $k_0 = \frac{k}{n}$ with n being the refractive index of the medium, and β and κ are the longitudinal and transverse components of the wavevector k, respectively, inside the film, as depicted in Figure 2. The angle θ, is the required angle of incidence of the wave.

This completes the mathematical model of the light wave in a planar waveguide, which leads us back to the original question of finding the angle of incidence θ of the wave to ensure TIR. β is the most interesting vector in this regard. It summarizes two of the very important characteristics of a wave in a medium. Firstly, because it is the longitudinal component of the wavevector, β contains the information about the wavelength of the wave. Secondly, it contains the propagation direction of the wave within the medium, which consequently gives us the angle of incidence θ. Now, in order to ensure the second condition for TIR, we need to find the corresponding βs. These specific values of βs are nominated to be the *eigenvalue of waveguides* since they contain all the information that is required to describe the behavior of the wave and the waveguide.

The electric and magnetic field equations (2) and (3) can be utilized along with their well-known continuous nature [18] to verify the following useful relationship, which is usually termed as the *eigenvalue equation* for β.

$$\tan(h\kappa_f) = \frac{\gamma_c + \gamma_s}{\kappa_f \left(1 - \frac{\gamma_c \gamma_s}{\kappa_f^2}\right)} \tag{7}$$

The good thing about this relationship is that it contains β along with all the physical characteristics of the planar waveguide, such as refractive indices and height. Thus, it can be used to evaluate the value of β in terms of the planar waveguide parameters. This way, we can tune these parameters in such a way that an appropriate value of β is attained that satisfies the second condition for TIR, i.e., $\sin^{-1}(\frac{\lambda\beta}{2\pi}) < critical\_angle$. All the values of β that satisfy the above conditions are usually termed as the TE modes in the planar waveguide.

In this paper, we present the higher-order-logic formalization of the electric and magnetic field equations for the planar wave guide, given in Equations (2) and (3), respectively. Then, based on these formal definitions, we present the formal verification of the eigenvalue equation, given in Equation (7). As outlined above, it is one of the most important relationships used for the analysis of planar waveguides, which makes its formal verification in a higher-order-logic theorem prover a significant step towards using them for conducting formal optical systems analysis.

4 Formalization of Electromagnetic Fields

In this section, we present the higher-order-logic formalization of the electric and magnetic fields for a planar waveguide in the TE mode. We also verify an expression for the magnetic field by differentiating the electric field expression.

The electric field, given in Equation (2), is a piecewise function, i.e., a function whose values are defined differently on disjoint subsets of its domain. Reasoning about the derivatives of piecewise functions in a theorem prover is a tedious task as it involves rewriting based on the classical definitions of differentiation and limit due to their domain dependant values. In order to facilitate such reasoning, we propose to formally define piecewise linear functions in terms of the *Heaviside step function* [1], which is sometimes also referred to as the *unit step function*. A Heaviside step function is a discontinuous, relatively simple piecewise, real-valued function that returns 1 for all strictly positive arguments, 0 for strictly negative arguments and its value at point 0 is usually $\frac{1}{2}$.

$$H(x) = \begin{cases} 0 & x < 0; \\ \frac{1}{2} & x = 0; \\ 1 & 0 < x. \end{cases} \tag{8}$$

By defining piecewise functions based on the Heaviside step function, we make their domain dependance implicit. It allows us to simply reason about the derivatives of piecewise functions using the differentiation properties of sum and products of functions. This way, we need to reason about the derivative of only one piecewise function, i.e., the Heaviside step function, from scratch and build upon these results to reason about the derivatives of all kinds of piecewise functions without utilizing the classical definitions. In this paper, we apply this approach to reason about the derivative of the electric field expression, given in Equation (2). The first step in this regard is to formalize the Heaviside step function as the following higher-order-logic function.

Definition 1: *Heaviside Step Function*

⊢ ∀ x. h_step x = if x=0 then $\frac{1}{2}$ else (if x<0 then 0 else 1)

Next, we formally verify that the derivative of `h_step` function for all values of its argument x, except 0, is equal to 0.

Theorem 1: *Derivative of Heaviside Step Function*

⊢ ∀ x. ¬(x = 0) ⇒ (deriv h_step x = 0)

where the HOL function `deriv` represents the derivative function [6] that accepts a real-valued function f and a differentiating variable x and returns df/dx. The proof of the above theorem is based on the classical definitions of differentiation and limit along with some simple arithmetic reasoning.

Now, the electric field of a planar waveguide, given in Equation (2), can be expressed in higher-order logic as the following function.

Definition 2: *Electric Field for the Planar Waveguide in TE mode*

```
⊢ ∀ b k n. gamma b k n = √(b² - k²n²)
⊢ ∀ b k n. kappa b k n = √(k²n² - b²)
⊢ ∀ A B C D n_c n_s n_f k_0 b h x.
   E_field A B C D n_c n_s n_f k_0 b h x =
     A e^((-(gamma b k_0 n_c))x) (h_step x) +
     (B cos((kappa b k_0 n_f) x) + C sin((kappa b k_0 n_f) x))
         (h_step (-x)) +
     (D e^((-(gamma b k_0 n_s))(x+h)) -
     (B cos((kappa b k_0 n_f) x) + C sin((kappa b k_0 n_f) x)))
         (h_step (-x - h))
```

The function `E_field` accepts the four amplitude coefficients `A, B, C` and `D`, the three refractive indices for the planar waveguide `n_c, n_s` and `n_f`, corresponding to the cover, substrate and the film regions, respectively, the vacuum wave vector `k_0`, the longitudinal component of the wave vector `b`, the height of the waveguide `h` and the variable `x` for the *x-axis*. It uses the function `gamma` to obtain the two attenuation coefficients in the cover and substrate as `(gamma b k_0 n_c)` and `(gamma b k_0 n_s)`, respectively, and the function `kappa` to model the transverse component of k in the guiding film as `(kappa b k_0 n_f)`. It also utilizes the Heaviside step function `h_step` thrice with appropriate arguments to model the three sub domains of the piecewise electric field for the planar waveguide, described by the above parameters, according to Equation (2). It is important to note that, rather than having the undefined values for the boundaries $x = 0$ and $x = -h$, as is the case in Equation (2), our formal definition assigns fixed values to these points. But, since we will be analyzing the amplitude coefficients under the continuity of electric and magnetic fields, these point values do not alter our results as will be seen in the next section.

Next, we formalize the magnetic field expression for the planar waveguide, given in Equation (3), using the functional definition of the derivative, `deriv`, given in [6], as follows.

Definition 3: *Magnetic Field for the Planar Waveguide*

```
⊢ ∀ omega mu A B C D n_c n_s n_f k_0 b h x.
    H_field omega mu A B C D n_c n_s n_f k_0 b h x =
    (1 / (omega mu)) deriv (λx. E_field A B C D n_c n_s n_f k_0 b h x) x
```

The function `H_field` accepts the frequency `omega` and the permeability of the medium `mu` besides the same parameters that have been used for defining the electric field of the planar waveguide in Definition 2. We have removed the *imaginary unit* part from the original definition, given in Equation (3), in the above definition for simplicity as our analysis is based on the amplitudes or absolute values of electric and magnetic fields and thus requires the real portion of the corresponding complex numbers only. However, if need arises, the imaginary part can be included in the analysis as well by utilizing the higher-order-logic formalization of complex numbers [7].

Definitions 2 and 3 can now be used to formally verify a relation for the magnetic field in a planar waveguide as follows:

Theorem 2: *Expression for the Magnetic Field*

```
⊢ ∀ omega mu A B C D n_c n_s n_f k_0 b h x.
  ¬(x = 0) ∧ ¬(x = -h) ⇒
    H_field omega mu A B C D n_c n_s n_f k_0 b h x = 1/(omega mu) (
    (-(gamma b k_0 n_c)) A e^(-(gamma b k_0 n_s))x (h_step x) +
    (kappa b k_0 n_f)
    (-B sin((kappa b k_0 n_f) x) + C cos((kappa b k_0 n_f) x))
          (h_step (-x)) +
    (gamma b k_0 n_s) (D e^((gamma b k_0 n_s))(x+h) -
         (kappa b k_0 n_f)
    (-B sin((kappa b k_0 n_f) x) + C cos((kappa b k_0 n_f) x)))
          (h_step (-x - h)))
```

This theorem can be verified by proving the derivatives of the three expressions found in the definition of the electric field and the derivative of the Heaviside step function, given in Theorem 2, along with basic differentiation properties of a product and sum of functions, formally verified in [6].

5 Verification of the Eigenvalue Equation

In this section, we build upon the formal definitions of electromagnetic field relations, formalized in the previous section, to formally verify the eigenvalue equation for the planar waveguide in the TE mode, given in Equation (7).

The main idea behind our analysis is to leverage upon the continuous nature of the electric and magnetic field functions. Like all other continuous functions, a continuous piecewise function f also approaches the value $f(x0)$ at any point $x = x0$ in its domain. This condition, when applied to the boundary points $x = 0$ and $x = -h$ of our piecewise functions for the electric and magnetic fields, `E_field` and `H_field`, respectively, yields very interesting results that allow us to express the amplitude coefficients B, C and D in terms of the amplitude coefficient A and then finally utilize these relationships to verify Equation (7).

Due to the piecewise nature of our electric and magnetic field functions, the above reasoning is based on the mathematical concept of right and left hand limits, sometime referred to as one-sided limits, which are the limits of a real-valued function taken as a point in their domain is approached from the right and from the left hand side of the real axis, respectively [2]. Therefore, the first step towards the formal verification of Equation (7) is the formalization of right hand limit in higher-order logic using its classical definition as follows:

Definition 4: *Limit from the Right*

```
⊢ ∀ f y0 x0. right_lim f y0 x0 =
    ∀ e. 0 < e ⇒ ∃d. 0 < d ∧
       ∀x. 0 < x - x0 ∧ x - x0 < d ⇒ abs(f x - y0) < e
```

The abs function is the HOL function for the absolute value of a real number. According to the above definition, the limit of a real valued function $f(x)$, as x tends to $x0$ from the right is $y0$, if for all strictly positive values e, there exists a number d such that for all x satisfying $x0 < x < x0 + d$, we have $|f(x) - y0| < e$. Similarly, the left hand limit can be formalized as follows:

Definition 5: *Limit from the Left*

```
⊢ ∀ f y0 x0. left_lim f y0 x0 =
    ∀ e. 0 < e ⇒ ∃d. 0 < d ∧
        ∀x. -d < x - x0 ∧ x - x0 < 0 ⇒ abs(f x - y0) < e
```

If the normal limit of a function exists at a point and is equal to $y0$ then both the right and left limits for that function are also well-defined for the same point and are both equal to $y0$. This is an important result for our analysis and thus we formally verify it in the HOL theorem prover as the following theorem.

Theorem 4: *Limit Implies Limit from the Right and Left*

```
⊢ ∀ f y0 x0. (f→y0)x0 ⇒ right_lim f y0 x0 ∧ left_lim f y0 x0
```

The assumption of the above theorem (f → y0)x0 represents the formalization of the normal limit of a function [6] and is True only if the function f approaches $y0$ at point $x = x0$. The proof of Theorem 4 is basically a re-writing of the definitions involved along with the properties of the absolute function. We also verified the uniqueness of both right and left hand limits as follows.

Theorem 5: *Limit from the Right is Unique*

```
⊢ ∀f y1 y2 x0. right_lim f y1 x0 ∧ right_lim f y2 x0 ⇒(y1=y2)
```

Theorem 6: *Limit from the Left is Unique*

```
⊢ ∀f y1 y2 x0. left_lim f y1 x0 ∧ left_lim f y2 x0 ⇒(y1=y2)
```

The proof of Theorem 5 is by contradiction, as it is not possible that a real-valued function gets as near as possible to two unequal points in its range for the same argument. We proceed with the proof by first assuming that $\neg(y1 = y2)$ and then rewriting the statement of Theorem 5 with the definition of the function right_lim. Next, the two assumptions are specialized for $e = \frac{|y1-y2|}{2}$ case. Now, the same x is chosen for both the assumptions in such a way that the conditions on x, i.e., $x0 < x < x0 + d$, for both of the assumptions are satisfied. One such x is $\frac{min\ d1\ d2}{2} + x0$, where $d1$ and $d2$ are the $d's$ for the two assumptions, respectively, and the function min returns the minimum value out of its two real number arguments. Thus, for such an x, the two given assumptions imply that $|fx - y1| < \frac{|y1-y2|}{2}$ and $|fx - y2| < \frac{|y1-y2|}{2}$, which leads to a contradiction in both of the cases when $y1 < y2$ and $y2 < y1$. Hence, our assumption $\neg(y1 = y2)$ cannot be True and $y1$ must be equal to $y2$, which concludes the proof of Theorem 5. Theorem 6 is also verified using similar reasoning.

The above infrastructure can now be utilized to formally verify the mathematical relationships between the amplitude coefficients. The relationship between the amplitude coefficients B and A can be formally stated as follows:

Theorem 7: $B = A$

```
⊢ ∀ A B C D n_c n_s n_f k_0 b h x. 0 < h ∧
    (∀x. (λx. E_field A B C D n_c n_s n_f k_0 b h x) contl x)
        ⇒ (B = A)
```

The first assumption ensures that h is always greater than 0 and is valid since h represents the height of the waveguide. Whereas, the HOL predicate (f contl x) [6], used in the above theorem, represents the relational form of a continuous function definition, which is True when the limit of the real-valued function f exists for all points x on the real line and is equal to $f(x)$. Thus, the corresponding assumption, in the above theorem, ensures that the function E_field is continuous on the $x - axis$ and its limit at the boundary points $x = 0$ and $x = -h$ is equal to the value of the function E_field at $x = 0$ and $x = -h$.

In order to verify Theorem 7, consider the boundary point $x = 0$, for which the value of the function E_field becomes $\frac{A+B}{2}$, according to Definition 2. Now, based on Theorem 5, the limit from the right at $x = 0$ for the function E_field is also going to be $\frac{A+B}{2}$. Next, we verified, using Definition 4 along with the properties of the exponential function [6], that the limit from the right for the function E_field at point $x = 0$ is in fact equal to A. The uniqueness of the right limit property, verified in Theorem 5, can now be used to verify that A must be equal to $\frac{A+B}{2}$ as they both represent the limit from the right for the same function at the same point. This result can be easily used to discharge our proof goal $A = B$, which concludes the proof for Theorem 7.

Next, we apply similar reasoning as above with the magnetic field relation for the planar waveguide, verified in Theorem 2, at point $x = 0$ to verify the following relationship between the amplitude coefficients C and A.

Theorem 8: $C = -A\frac{\gamma_c}{\kappa_f}$

```
⊢ ∀ omega mu A B C D n_c n_s n_f k_0 b h x.(0 < h)∧(0 < mu)∧
(0 < omega)∧(b < k_0 n_f)∧(k_0 n_s < b)∧(0 < n_s)∧(0 < k_0)∧
(∀x.λx.H_field omega mu A B C D n_c n_s n_f k_0 b h x) contl x)
        ⇒ (C = -A (gamma b k_0 n_c)/(kappa b k_0 n_f))
```

The additional assumptions besides, $0 < h$, used in the above theorem, ensure that the values of the functions gamma and kappa are positive real numbers and do not attain an imaginary complex number value, according to their definitions, given in Section 3. Again based on the continuity of the magnetic field H_field assumption, we know that its limit at point 0 is equal to the value of H_field at $x = 0$, say H_0. It is important to note that the value of H_0 cannot be obtained from the expression for the H_field, given in Theorem 2. Therefore, we cannot reason about its precise value but based on the continuity of H_field, we do know that it exists. This implies that the limit from right and left for this function would be also equal to H_0, according to Theorem 4. Next, we verified that limits from right and left for the magnetic field function H_field, given in Theorem 2, at point $x = 0$ are $\frac{-A(gamma\ b\ k\_0\ n\_c)}{om\ mu}$ and $\frac{C(kappa\ b\ k\_0\ n\_f)}{om\ mu}$ using Definitions 4 and 5, respectively. This leads to the verification of Theorem 8, since we already know that these two limit values are equal to H_0, using the uniqueness of limits from right and left, verified in Theorems 5 and 6.

Now, using similar reasoning as above and applying continuity of `E_field` and `H_field` at $x = -h$, we verified the following two relations to express the amplitude coefficient D in terms of the amplitude coefficients B and C.

Theorem 9: $D = B\cos(\kappa_f h) - C\sin(\kappa_f h)$

```
  ⊢ ∀ A B C D n_c n_s n_f k_0 b h x. 0 < h ∧
      (∀x. (λx. E_field A B C D n_c n_s n_f k_0 b h x) contl x)
⇒ (D = B(cos((kappa b k_0 n_f) h)) - C(sin((kappa b k_0 n_f) h)))
```

Theorem 10: $D = \kappa_f \frac{B\sin(\kappa_f h)+C\cos(\kappa_f h)}{\gamma_s}$

```
  ⊢ ∀ omega mu A B C D n_c n_s n_f k_0 b h x. (0 < h) ∧
  (0 < mu)∧(0 < omega)∧(k_0 n_s < b)∧(0 < n_s)∧(0 < k_0) ∧
  (∀x.(λx.H_field omega mu A B C D n_c n_s n_f k_0 b h x)contl x)
```

$\Rightarrow$(D=(kappa b k_0 n_f) $\frac{\text{(B(sin((kappa b k\_0 n\_f) h))+C(cos((kappa b k\_0 n\_f) h))))}}{\text{(gamma b k\_0 n\_s)}}$

The above theorems allows us to reach an alternate expression for `E_field` in terms of only A, which is the amplitude of the electric field at $x = 0$. This relationship is very useful for plotting the mode profiles of guided modes [18].

The right-hand sides of the conclusions of Theorems 9 and 10 can now equated together, since both are equal to D, and the amplitudes coefficients B and C can be expressed in terms of A, using Theorems 7 and 8, respectively, to formally verify the desired relationship for evaluating the eigenvalues of the planar waveguide, given in Equation (7), as the following theorem.

Theorem 11: *Eigenvalue Equation*

```
  ⊢ ∀ omega mu A B C D n_c n_s n_f k_0 b h x. (0 < A) ∧
  (0 < h) ∧ (0 < mu) ∧ (0 < omega) ∧
  (b < k_0 n_f) ∧ (k_0 n_s < b) ∧ (0 < n_s) ∧ (0 < k_0) ∧
  ¬((kappa b k_0 n_f)^2 = (gamma b k_0 n_c) (gamma b k_0 n_s)) ∧
  (∀x.(λx.E_field A B C D n_c n_s n_f k_0 b h x)contl x) ∧
  (∀x.(λx.H_field omega mu A B C D n_c n_s n_f k_0 b h x)contl x)
    ⇒ (tan((kappa b k_0 n_f) h) =
```

$$\frac{\text{(gamma b k\_0 n\_c)+(gamma b k\_0 n\_s)}}{\text{(kappa b k\_0 n\_f)}\left[1-\frac{\text{(gamma b k\_0 n\_c)(gamma b k\_0 n\_s)}}{\text{(kappa b k\_0 n\_f)}^2}\right]}\Bigg)$$

Due to the inherent soundness of the theorem proving approach, our verification results exactly matched the paper-and-pencil analysis counterparts for the eigenvalue equation, as conducted in [18], and thus can be termed as 100% precise. Interestingly, the assumption `¬((kappa b k_0 n_f)^2 = (gamma b k_0 n_c) (gamma b k_0 n_s))`, without which the eigenvalues are undefined, was found to be missing in [18]. This fact clearly demonstrates the strength of formal methods based analysis as it allowed us to highlight this corner case, which if ignored could lead to the invalidation of the whole eigenvalue analysis.

The verification results, given in this section, heavily relied upon real analysis and thus the useful theorems available in the HOL real analysis theories [6] proved to be a great asset in this exercise. The verification task took around 2500 lines of HOL code and approximately 100 man-hours.

6 Application: Planar Asymmetric Waveguide

In this section, we demonstrate the effectiveness of Theorem 11 in analyzing the eigenvalues of a planar asymmetric waveguide [18]. The waveguide is characterized by a guiding index n_f of 1.50, the substrate index n_s of 1.45 and the cover index n_c of 1.40. The thickness of the guiding layer h is $5\mu m$. The goal is to determine the allowable values of β for this structure, assuming that the wavelength λ of $1\mu m$ is used to excite the waveguide.

In order to obtain the allowable values of β from Theorem 11, we rewrite it with the definition of the function gamma, replace the term (kappa b k_0 n_f) with k_f and express the variable b, which represents β in our Theorems, in terms of k_f as $\sqrt{(\texttt{k\_0})^2(\texttt{n\_f})^2 - (\texttt{k\_f})^2}$, using the definition of kappa, to obtain the following alternate relationship.

Theorem 12: *Alternate form of Eigenvalue Equation*

$$\vdash \forall \texttt{ omega mu A B C D n\_c n\_s n\_f k\_0 b h x. } (0 < \texttt{A}) \wedge$$
$$(0 < \texttt{h}) \wedge (0 < \texttt{mu}) \wedge (0 < \texttt{omega}) \wedge$$
$$(\texttt{b} < \texttt{k\_0 n\_f}) \wedge (\texttt{k\_0 n\_s} < \texttt{b}) \wedge (0 < \texttt{n\_s}) \wedge (0 < \texttt{k\_0}) \wedge$$
$$\neg((\texttt{kappa b k\_0 n\_f})^2 = (\texttt{gamma b k\_0 n\_c})\ (\texttt{gamma b k\_0 n\_s})) \wedge$$
$$(\forall \texttt{x}.(\lambda \texttt{x.E\_field A B C D n\_c n\_s n\_f k\_0 b h x})\texttt{contl x}) \wedge$$
$$(\forall \texttt{x}.(\lambda \texttt{x.H\_field omega mu A B C D n\_c n\_s n\_f k\_0 b h x})\texttt{contl x})$$
$$\Rightarrow \Big(\tan(\texttt{k\_f h}) = \frac{\sqrt{((\texttt{k\_0})^2(\texttt{n\_f})^2 - (\texttt{k\_f})^2) - (\texttt{k\_0})^2(\texttt{n\_c})^2} + \sqrt{((\texttt{k\_0})^2(\texttt{n\_f})^2 - (\texttt{k\_f})^2) - (\texttt{k\_0})^2(\texttt{n\_s})^2}}{\texttt{k\_f}\left[1 - \frac{(\sqrt{((\texttt{k\_0})^2(\texttt{n\_f})^2 - (\texttt{k\_f})^2) - (\texttt{k\_0})^2(\texttt{n\_c})^2})(\sqrt{((\texttt{k\_0})^2(\texttt{n\_f})^2 - (\texttt{k\_f})^2) - (\texttt{k\_0})^2(\texttt{n\_s})^2})}{(\texttt{k\_f})^2}\right]}\Big)$$

All the quantities in the conclusion of the above theorem are known except k_f, since k_0 can be expressed in terms of the wavelength that is used to excite the waveguide, as outlined in Section 3. Though, getting a closed form solution for k_f is not possible from the above equation. Therefore, we propose to use a computer algebra system to solve for the value of k_f. Using Mathematica, the first four eigenvalues of k_f were found to be 5497.16, 10963.2, 16351 and 21545 cm^{-1}. These values can then be used to calculate the desired eigenvalues for b according to the following relationship $\texttt{b} = \sqrt{(\texttt{k\_0})^2(\texttt{n\_f})^2 - (\texttt{k\_f})^2}$, and were found to be 94087, 93608, 92819 and 91752 cm^{-1}.

Hypothetically the above analysis can be divided into two parts. The first part covers the analysis starting from the electromagnetic wave equations, with the given parameters, up to the point where we obtain the alternate form of eigenvalue equation, given in Theorem 12. The second part is concerned with the actual computation of eigenvalues from Theorem 12. The first part of the above analysis was completely formal and thus 100% precise, since it was done using the HOL theorem prover. The proof script for this theorem was less than 100 lines long, which clearly demonstrates the effectiveness of our work, as it was mainly due the the availability of Theorem 11 that we were able to tackle this kind of a verification problem with such a minimal effort. The second part of the analysis cannot be handled in HOL, because of the involvement of a transcendental equation for which a closed form solution for `k_f` does not exist. For this part, we utilized Mathematica and obtained the desired eigenvalues. To the best of our knowledge, no other approach based on simulation, numerical methods or computer algebra systems, can provide 100% precision and soundness in the results like the proposed approach for the first part of the analysis. Whereas, in the second part, we have used a computer algebra system, which is the best option available, in terms of precision, for this kind of analysis. Other approaches used for the second part include graphical or numerical methods, which cannot compete with computer algebra systems in precision. Thus, as far as the whole analysis is concerned, the proposed method offers the most precise solution.

7 Conclusions

This paper presents the formal analysis of planar optical waveguides using a higher-order-logic theorem prover. Planar optical waveguides are simple, yet widely used optical structures and not only find their applications in wave guiding, but also in coupling, switching, splitting, multiplexing and de-multiplexing of optical signals. Hence, their formal analysis paves the way to the formal analysis of many other optical systems as well. Since the analysis is done in a theorem prover, the results can be termed as 100% precise, which is a novelty that cannot be achieved by any other computer based optical analysis framework.

We mainly present the formalization of the electromagnetic field equations for a planar waveguide in the TE mode. These definitions are then utilized to formally reason about the eigenvalue equation, which plays a vital role in the design of planar waveguides for various engineering and other scientific domains. To illustrate the effectiveness and utilization of the formally verified eigenvalue equation, we used to reason about the eigenvalues of a planar asymmetric waveguide. To the best of our knowledge, this is the first time that a formal approach has been proposed for the analysis of optical systems.

The successful handling of the planar waveguide analysis clearly demonstrates the effectiveness and applicability of higher-order-logic theorem proving for analyzing optical systems. Some of the interesting future directions in this novel domain include the verification of the eigenvalue equation for the planar waveguide in the TM mode, which is very similar to the analysis presented in this paper, and the analysis of couplers that represent two or more optical devices linked together with an optical coupling relation, which can be done by building on top of the results presented in this paper along with formalizing the couple mode theory [8] in higher-order logic. Besides these, many saftey-critical planar waveguide applications can be formally analyzed including biosensors [23] or medical imaging [15] by building on top of our results.

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