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Formal verification of stability and chaos in periodic optical systems

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ABSTRACT

Optical systems are widely used in a wide range of safety, cost and mission-critical applications including biomedical devices and high-speed communication networks. Therefore, modeling and verification of such high-consequence systems is crucial for both theoretical and application viewpoints. In this paper, we propose a formal methods based approach to model and verify the properties of periodic optical systems which allow a cyclic passage of light through a sequence of optical components. We focus on two important properties namely stability and chaotic map generation which ensure the confinement of light and chaos generation, respectively. We use higher-order logic as a specification and reasoning framework and develop a library of necessary notions of periodic optical systems. Consequently, we demonstrate the utilization and effectiveness of our development by a couple of case studies: a Fabry Pérot resonator with fiber-rod lens and a phase-conjugated ring resonator.

1. Introduction

In the last few decades, the applications of optics have emerged in remote sensing [47], biomedical imaging [12], communications [10], high-speed computing [33] and aerospace [31] to name just a few. Interestingly, the 68th Session of the UN General Assembly (in December 2013) proclaimed 2015 as the *International Year of Light and Light-based Technologies* (IYL 2015) [2]. The main purpose of celebrating IYL was to consider that the applications of light science and technology are vital for existing and future advances in energy, information and communications, fiber optics, agriculture, mining, astronomy, architecture, archeology, entertainment, art and culture, as well as many other industries and services [3]. As a result, optics being the mainstream light based technology is expected to gain more awareness about its problem-solving potential among international policy-makers and stake-holders.

The designing of different optical systems depends heavily on the modeling choices for the light and optical components (e.g., mirrors and lenses). In fact, light can be modeled at different levels of abstraction such as geometrical, wave, electromagnetic and quantum optics. Geometrical optics characterizes light as a set of straight lines or beams that linearly traverse through an optical system [9]. Wave optics [43] and electromagnetic optics [43] describe the scalar and vectorial wave nature of light, respectively. On the other hand, quantum optics [13] characterizes light as a stream of photons and helps to tackle situations where it is necessary to consider both wave-like and particle-like behaviors of light. In general, each of these theories has been used to model different aspects of the same or different optical components. For example, a

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phase-conjugate mirror [18] can be modeled using ray, electromagnetic and quantum optics. The application of each theory is dependent on the type of systems and properties of interest which need to be analyzed. The main focus of this paper is periodic optical systems in which a ray passes through a sequence of optical components that repeat periodically along the axis [39]. In optical engineering, such systems are mainly used to study the paths of light rays bouncing back and forth between pairs of mirrors. This allows a convenient approach to deduce conditions for the *stability* [39] and *chaos generation* [4] which ensure that a ray never deviates more than a certain finite distance from the axis and a ray follows some chaotic map, respectively.

The design flow of optical systems involves the physical modeling of optical components, analysis, and prototyping. This process is always subject to time and cost constraints due to the increased deployment of optical devices in various industries. Therefore, a significant portion of time is spent in the analysis and verification of system models to find bugs in the design process prior to the manufacturing of the actual system. Minor bugs in optical systems can lead to fatal consequences such as the loss of human lives because of their use in surgeries and high precision biomedical devices, or financial loss because of their use in high budget space missions. The impact of optical systems failure is more critical in applications that are directly linked to safety issues such as in aerospace as compared to telecommunication where failures can lead to safety problems through some secondary events. An example of such a critical application is Boeing F/A-18E, for which the mission management system is linked using an optical network [46]. Considering these facts, it is very important to build a framework for the analysis of optical systems which is both accurate and scalable.

Formal methods [17] are computer based reasoning techniques which allow accurate and precise analysis and thus have the potential to overcome the limitations of accuracy, found in traditional approaches. The main idea behind formal methods based analysis of systems is to develop a mathematical model for the given system and analyze this model using computer-based mathematical reasoning, which in turn increases the chances for catching subtle but critical design errors that are often ignored by traditional techniques. The two major formal methods techniques are *model checking* and *theorem proving* (a brief overview of other formal methods techniques can be found in [17]). Model checking is an automated verification technique for systems that can be expressed as finite-state machines. On the other hand, higher-order-logic (HOL) theorem proving is an interactive verification technique, which is mainly based on the notion of formal proofs in a logic (e.g., propositional logic, first-order logic or higher-order logic). Due to the involvement of multivariate analysis and complex-valued parameters, model checking cannot be used to analyze hardware aspects of optical systems. However, HOL theorem proving can be applied in optics due to its higher expressibility and the availability of some well-developed theorem proving systems. Nowadays, the use of formal methods for high risk and safety-critical systems is recommended in different standards like the electronics IEC 61508 [22], DO178-C [19] for aviation and ISO 26262 [20] for automotive systems.

In order to build an accurate and reliable optical system analysis framework, a project[1] [5] was initiated in our research group at Concordia University in 2009. The main scope of this project includes the higher-order logic formalization of different theories of optics which provides the basis to conduct more accurate analysis than traditional paper-and-pencil based proofs (e.g., [4,29]), numerical simulation (e.g., [32,41,23]) and computer algebra systems (CAS) [30]. So far, the formalization of ray optics [38], electromagnetic optics [21] and quantum optics [24] has been implemented in the HOL Light [16] proof assistant. In this paper, however, we aim at building a framework to verify stability and chaos generation in periodic optical systems based on the ray-transfer-matrix approach [38]. In particular, we define new data-structure to model a wide class of optical interfaces including phase-conjugate mirrors and an unknown element which was not possible in our earlier formalization of ray optics [38,37]. We then formalize the notion of periodic optical systems and associated ray behavior to consider an infinite number of round-trips of light rays. This includes the development of a formal stability analysis framework and the formalization of chaotic maps inside periodic optical systems. This development allows us to conduct the stability verification of the Fabry Pérot resonator and the verification of two-dimensional chaotic map generation inside a phase-conjugate ring resonator. The source code of our formalization is available for download [44] and can be used by other researchers and optical engineers for further developments.

The rest of the paper is organized as follows: Section 2 introduces some fundamentals of interactive theorem proving and the HOL Light proof assistant. We develop the details of our modeling choices of optical systems and light rays along with their HOL Light implementation in Section 3. We present the formalization of periodic optical systems in Section 4. We then present the formalization of stability and chaos in Section 5. In order to demonstrate the practical effectiveness and the use of our work, we describe in Section 6 the analysis of two real-world optical resonators: the Fabry–Pérot resonator with fiber-rod lens and a ring resonator. We discuss the challenging aspects of our work and gained experience in Section 7. Finally, Section 8 concludes the paper and highlights some future directions.

2. Theorem proving and HOL light

Theorem proving is a widely used formal methods technique which is concerned with the construction of mathematical theorems by a computer program (called *theorem prover or proof assistant*) [15]. Theorem proving systems have mainly been employed to verify generic properties of a wide class of software and hardware systems. For example, a hardware de-

[1] http://hvg.ece.concordia.ca/projects/optics/.

signer can prove different properties of a digital circuit by describing its behavior by some predicates and applying Boolean algebra. Similarly, a mathematician can prove the transitivity of real number ordering using fundamental axioms of real numbers theory. These properties are described as theorems in a particular logic such as propositional logic, first-order logic or higher-order logic, depending upon the expressibility requirements. For example, the use of higher-order logic is advantageous over first-order logic in terms of the availability of additional quantifiers and its high expressiveness. Moreover, higher-order logic is expressive enough to describe almost all known concepts from Mathematics including topological spaces, real numbers, multivariate calculus and higher transcendental functions. Once such a mathematical theory is expressed inside a theorem prover, we say that it is formalized.

HOL Light (an acronym for a lightweight implementation of Higher-Order Logic) [16] is an interactive theorem proving environment for the construction of mathematical proofs. The main implementation of HOL Light is done in Objective CAML (OCaml), which is a functional programming language originally developed to automate mathematical proofs [1]. The main components of the logical kernel of HOL Light (approximately 400 lines of OCaml code) are its types, terms, theorems, rules of inference, and axioms. We present a brief overview [14] of each of them as follows:

- **Types:** The foundation of HOL Light is based on the notion of types and there are only two primitive types, i.e., the Boolean type (`:bool`) and an infinite type (`:ind`). The other types are generated from type variables $:x$; $:y$; ... and primitive types (Boolean or infinite) using an arrow $\rightarrow$. For example, `:bool` and `:bool` $\rightarrow x$ represent types. Note that a colon (:) is used to specify the corresponding type.
- **Terms:** The terms are the basic objects of HOL Light and their syntax is based on the λ-calculus. We can use λ-terms, also called lambda abstractions, e.g., $\lambda x. f(x)$ represents a function which takes x and returns $f(x)$. The collection of terms is constructed from variables x; y; ... and constants 0; 1; ... using the λ-abstraction ($\lambda x.t$). Each term has a type which can be represented by the notation $x : A$, i.e., the type of term x is A.
- **Inference Rules:** Inference rules are procedures for deriving new theorems, which are represented as OCaml functions. HOL Light has ten basic inference rules and a mechanism for defining new constants and types. Some of the inference rules are the reflexivity of equality, the transitivity of equality and the fact that equal functions applied to equal arguments are equal [16].
- **Axioms:** The kernel of HOL Light has three mathematical axioms: 1) the axiom of extensionality which states that a function is determined by the values that it takes on all inputs; 2) the axiom of infinity which states that the type `:ind` is not finite; and 3) the axiom of choice which states that we can choose a term that satisfies a predicate.
- **Theorems:** A theorem is a formalized statement that is either an axiom or can be deduced from already verified theorems by inference rules. A theorem consists of a finite set Ω of Boolean terms called the assumptions and a Boolean term S called the conclusion. For example, "$\forall x. x \neq 0 \Rightarrow \frac{x}{x} = 1$" represents a theorem in HOL Light.

A HOL Light theory consists of a set of types, constants, definitions, axioms and theorems. HOL theories are organized in a hierarchical fashion and theories can inherit the types, constants, definitions and theorems of other theories as their parents. Proofs in HOL Light are based on the concepts of tactics and tacticals that break goals into simpler subgoals. There are many automatic proof procedures and proof assistants available in HOL Light which help the user in directing the proof to the end [16]. In this paper, we try to use as much as possible standard mathematical notations rather than HOL Light ASCII code. In HOL Light, the datatypes for real numbers, positive integers and complex numbers are represented by `real`, `num` and `complex`, respectively. The notation `A**B` represents the matrix–matrix or matrix–vector multiplication.

3. Optical systems modeling and formalization using ray optics

Ray optics describes the propagation of light as rays through different interfaces and mediums. The main principle of ray optics is based on some postulates which can be summed up as follows: Light travels in the form of rays emitted by a source; an optical medium is characterized by its refractive index; light rays follow the Fermat's principle of least time [35]. Generally, the main components of optical systems are lenses, mirrors and a propagation medium which is either a free space or some material such as glass. These components are usually centered about an optical axis, around which rays travel at small inclinations (angle with the optical axis). Such rays are called *paraxial rays* and this assumption provides the basis of *paraxial optics* which is the simplest framework of geometrical optics. When a ray passes through optical components, it undergoes *translation*, *refraction* or *reflection*. In translation, the ray simply travels in a straight line from one component to the next and we only need to know the thickness of the translation. On the other hand, refraction takes place at the boundary of two regions with different refractive indices and the ray obeys the law of refraction, called *Paraxial Snell's law* [35]. Similarly, a ray follows the law of reflection at the boundary of a reflective interface (e.g., mirror).

Optical systems are hierarchically structured with optical interfaces (e.g., spherical and plane) as the base layer, optical components and systems forms next two layers. In order to correctly specify the physical behavior of the overall systems, we need to describe the ray behavior at each level of the optical systems. Any optical component can be modeled as a matrix in geometrical optics, considering the rotational symmetry and fixing a reference axis or optical axis. A single ray can be characterized by a vector depending upon the angle and vertical distance of the ray from the optical axis. Moreover, tracing a ray through an optical system can be done by computing the whole system matrix which is indeed a composition of individual matrices of corresponding optical components. This matrix formalism (called *ray-transfer matrices*) of ray optics

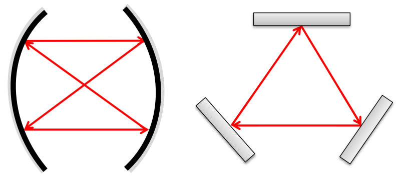

Fig. 1. Optical resonators.

provides an efficient, scalable and systematic analysis of real-world complex optical and laser systems. This is because of the fact that each optical component can be described as a 2 × 2 matrix and all linear algebraic properties can be used in the analysis of optical systems. For example, the general optical system with an input and output ray vector can be described as follows:

$$\begin{bmatrix} y_{n+1} \\ \theta_{n+1} \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} y_n \\ \theta_n \end{bmatrix}$$

where y_i and θ_i represent ray height and ray angle with optical axis, respectively. The parameters A, B, C and D are the components of the ray-transfer matrix which relates the input ray angle and ray height to the corresponding output ones. Finally, if we have an optical system consisting of N optical components, then we can trace the input ray R_0 through all optical components using the composition of matrices of each optical component as follows:

$$R_n = (M_k \cdot M_{k-1} \cdots M_1) \cdot R_0 \tag{1}$$

where M_i represents the ith component of the system. We can write $R_n = M_s R_0$, where $M_s = \prod_{i=k}^{1} M_i$. Here, R_n is the output ray and R_0 is the input ray.

Periodic optical systems are structured in such a way that light remains confined inside a closed structure called optical cavity. These systems are also called optical *resonators*, which are considered fundamental building-blocks for a wide class of optical systems with applications in lasers [39], optical bio-sensors [8], refractometry [40] and wavelength division multiplexing-passive optical network (WDM-PON) systems [34]. An optical resonator usually consists of mirrors or lenses which are configured in such a way that the beam of light is confined in a closed path as shown in Fig. 1. In general, resonators differ by their geometry and components (interfaces and mirrors) used in their design. Optical resonators are broadly classified as stable or unstable. The stability analysis identifies geometric constraints of the optical components which ensure that light remains inside the resonator. Both stable and unstable resonators have diverse applications, e.g., stable resonators are used in the measurement of refractive index of cancer cells [40], whereas unstable resonators are used in the laser oscillators for high energy applications [39]. In the last few decades, there is an increasing interest in studying the chaotic behavior of optical resonators [4]. In fact, chaotic optical resonators have been used for secure and high-speed transmission of messages in optical-fiber networks [6] and efficient light energy storage [36].

3.1. Formalization of optical systems

Ray optics explains the behavior of light when it passes through a free space and interacts with different optical interfaces. We can model free space by a pair of real numbers (n, d), which are essentially the refractive index and the total width, as shown in Fig. 2(a). We consider five optical interfaces, i.e., plane-transmitted, plane-reflected, spherical-transmitted, spherical-reflected and phase conjugated mirror (PCM) as shown in Fig. 2. In geometrical optics, we describe a spherical interface by its radius of curvature (R). Note that, among curved mirrors, parabolic mirrors also play an important role in modeling optical systems. However, in the regime of geometrical (or paraxial optics), parabolic mirrors can be substituted by spherical mirrors, which are much easier to manufacture [28].

In HOL Light, we can use available types (e.g., `real`, `complex`, etc.) to abbreviate new types. We use this feature to define a type abbreviation for a free space as follows:

```
free_space  = real # real
```

where `real # real` corresponds to a pair of real numbers ($\mathbb{R} \times \mathbb{R}$).

In many situations, it is convenient to define new types in addition to the ones which are already available in HOL Light theories. One common way is to use enumerated types, where one gives an exhaustive list of members of the new type. In our formalization, we package different optical interfaces in one enumerated type definition to simplify the formal reasoning

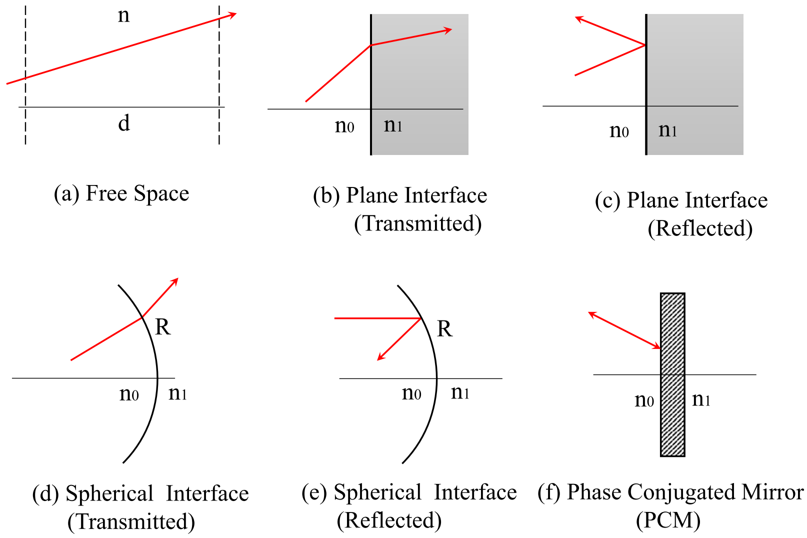

Fig. 2. Optical interfaces.

process, as follows:

```
interface = plane_transmitted |
             plane_reflected |
             spherical_transmitted real |
             spherical_reflected real |
             pcm |
             unknown real real real real
```

Note that we also include `unknown` as part of the type `interface`. The main motivation of considering an unknown element is to tackle the cases where a full description of an optical interface is not known in advance. Moreover, we parameterize the `unknown` element by four real-numbers to consider some prior information such as radius of curvature, aperture, etc. Note that this datatype can easily be extended to many other optical interfaces if needed.

In HOL Light, type definition considers the members of the type as constructors and it returns a pair of theorems, one for induction, one for recursion, as follows:

Theorem 1 *(Interface induction).*

```
⊢ ∀ P.  P plane_transmitted ∧
        P plane_reflected ∧
  (∀ a. P (spherical_transmitted a)) ∧
  (∀ a. P (spherical_reflected a)) ∧
        P pcm ∧
  (∀ a0 a1 a2 a3. P (unknown a0 a1 a2 a3))
   ⇒ (∀x. P x)
```

This theorem states that a property `P` holds for all objects of type `interface` if it holds for all the members of the type `interface`. In other words, Theorem 1 expresses that the type `interface` does not contain any other elements than those listed in its definition.

Theorem 2 *(Interface recursion).*

```
⊢∀ f0 f1 f2 f3 f4 f5.
∃ fn.fn plane_transmitted = f0 ∧
     fn plane_reflected = f1 ∧
(∀ a.fn (spherical_transmitted a) = f2 a)∧
(∀ a.fn (spherical_reflected a) = f3 a) ∧
     fn pcm = f4 ∧
(∀ a0 a1 a2 a3.fn (unknown a0 a1 a2 a3) = f5 a0 a1 a2 a3)
```

This theorem states that given any five parameters f0, f1, f3, f4 and f5, we can always define a function mapping from interfaces to these parameters, respectively.

We model an optical component as a pair (fs,i), i.e., free space, and an optical interface. This modeling choice is based on the physical behavior as we always consider a free space between two consecutive optical interfaces. Consequently, we define an optical system as a list of optical components followed by a free space. Following are the corresponding formal type abbreviations for optical component and optical system:

```
optical_component = free_space # interface
optical_system = optical_component list # free_space
```

Note that the use of a list in the optical_system type provides the facility to consider a system with any number of optical components. We formally verify some theorems which state that we can decompose the types free_space, optical_component and optical_system into their constituent components.

Theorem 3 *(Forall (∀) theorems for type abbreviations).*

```
⊢ ∀P. (∀(fs:free_space). P fs) ⇔ (∀n d. P (n,d))
⊢ ∀P. (∀(c:optical_component). P c) ⇔ (∀fs i. P (fs,i))
⊢ ∀P. (∀(os:optical_system). P os) ⇔ (∀cs fs. P (cs,fs))
```

Example. A thick lens is represented as the composition of two transmitting spherical interfaces separated by a nonzero distance in between both interfaces. Consider an arbitrary representation of thick lens which consists of three free spaces ((n_1,d), (n_2,d) and (n_1,d)) and two spherical interfaces with radius of curvatures R_1 and R_2. We can formalize this physical description of thick lens in HOL Light as follows:

```
⊢∀ R1 R2 n1 n2 d.
 thick_lens (R1,R2,n1,n2,d) =
 ([(n1,d),spherical_transmitted R1;(n2,d),spherical_transmitted R2],(n1,d))
```

Note that the type of the function thick_lens is $(\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}) \to$ optical_system. The parameter n_1 represents the refractive index before and after the first and the second interface. Whereas n_2 represents the refractive index between the two spherical interfaces which have the radius of curvatures R_1 and R_2, respectively.

A value of type free_space does represent a real space only if the refractive index is greater than zero. In addition, in order to have a fixed order in the representation of an optical system, we impose that the distance of an optical interface relative to the previous interface is greater or equal to zero. We encode this requirement in the following predicate:

Definition 1 *(Valid free space).*

```
⊢def is_valid_free_space (n,d) ⇔ 0 < n ∧ 0 ≤ d
```

where the type of is_valid_free_space is : free_space → bool.

We also need to assert the validity of a value of type interface by ensuring that the radius of curvature of spherical interfaces is never equal to zero. This yields the following predicate:

Definition 2 *(Valid optical interface).*

```
⊢def (is_valid_interface plane_transmitted ⇔ T) ∧
     (is_valid_interface plane_reflected ⇔ T) ∧
     (is_valid_interface (spherical_transmitted R) ⇔ ¬(0 = R)) ∧
     (is_valid_interface (spherical_reflected R) ⇔ ¬(0 = R)) ∧
     (is_valid_interface pcm ⇔ T) ∧
     (is_valid_interface (unknown a b c d) ⇔ T)
```

where the type of is_valid_interface is : interface → bool.

We now assert the validity of an optical system structure by ensuring the validity of every optical component in a system, as follows:

Definition 3 *(Valid optical component).*

$\vdash_{def}$
```
∀ fs i.
  is_valid_optical_component (fs,i) ⇔
  is_valid_free_space fs ∧ is_valid_interface i
```

Definition 4 *(Valid optical system).*

$\vdash_{def}$
```
∀ cs fs.
  is_valid_optical_system (cs,fs) ⇔
  ALL is_valid_optical_component cs ∧ is_valid_free_space fs
```

where `ALL` is a HOL Light library function which checks that a predicate holds for all the elements of a list.

We conclude our formalization of optical systems by defining a function to retrieve the refractive index of the first free space in an optical system:

Definition 5 *(Head index).*

$\vdash_{def}$
```
head_index ([],n,d) = n ∧
  head_index (CONS ((n,d),i) cs,nt,dt) = n
```

where `[]` represents an empty list of optical components.

3.2. Formalization of light rays

One of the important requirements for the formal analysis of optical systems is the formalization of rays which can specify the physical behavior of the light when it passes through an optical system. We only model the points where it hits an optical interface (instead of modeling all the points constituting the ray). So it is sufficient to just provide the distance of each of these hitting points to the optical axis and the angle taken by the ray at these points. Consequently, we should have a list of such pairs (*distance*, *angle*) for every component of a system. In addition, the same information should be provided for the source of the ray. For the sake of simplicity, we define a type for a pair (*distance*, *angle*) as `ray_at_point` as follows:

```
ray_at_point = real # real
ray = ray_at_point # ray_at_point #
     (ray_at_point # ray_at_point) list
```

where the first `ray_at_point` is the pair (*distance*, *angle*) for the source of the ray, the second one is the one after the first free space, and the list of `ray_at_point` pairs represents the same information for the interfaces and free spaces at every hitting point of an optical system.

Once again, we specify what is a valid ray by using some predicates. First of all, we define the behavior of a ray when it is traveling through a free space. This requires the position and orientation of the ray at the previous and current points of observation, and the free space itself. This is shown in Fig. 2(a).

Definition 6 *(Behavior of a ray in free space).*

$\vdash_{def}$ is_valid_ray_in_free_space (y_0,θ_0) (y_1,θ_1) ((n,d):free_space) $\Leftrightarrow$
$y_1 = y_0 + d * \theta_0 \wedge \theta_0 = \theta_1$

We next define the valid behavior of a ray when hitting a particular interface. This requires the position and orientation of the ray at the previous and current interfaces, and the refractive indices before and after the component. Then the predicate is defined by case analysis on the interface type as follows:

Definition 7 *(Behavior of a ray at given interface).*

$\vdash_{def}$ C1:(is_valid_ray_at_interface (y_0,θ_0) (y_1,θ_1) n_0 n_1
plane_transmitted $\Leftrightarrow y_1 = y_0 \wedge n_0 * \theta_0 = n_1 * \theta_1$) $\wedge$
C2:(is_valid_ray_at_interface (y_0,θ_0) $(y1,\theta_1)$ n_0 n_1
(spherical_transmitted R) $\Leftrightarrow$ let $\phi_i = \theta_0 + \frac{y_1}{R}$ and $\phi_t = \theta_1 + \frac{y_1}{R}$ in

```
    y1 = y0 ∧ n0 * ϕi = n1 * ϕt) ∧
C3:(is_valid_ray_at_interface (y0,θ0) (y1,θ1) n0 n1
     plane_reflected ⇔ y1 = y0 ∧ n0 * θ0 =  n0 * θ1) ∧
C4:(is_valid_ray_at_interface (y0,θ0) (y1,θ1) n0 n1
    (spherical_reflected R) ⇔ let ϕi = y1/R - θ0 in y1 = y0 ∧
    θ1 = -(θ0 +  2 * ϕi))∧
C5:(is_valid_ray_at_interface (y0,θ0) (y1,θ1) n0 n1 pcm
    ⇔ y1 = y0 ∧ θ1 = -θ0) ∧
C6:(is_valid_ray_at_interface (y0,θ0) (y1,θ1) n0 n1
    (unknown a b c d) ⇔ y1 = a*y0 + b*θ0 ∧ θ1 = c*y0 + d*θ0)
```

where each case C1-C6 states some basic geometrical facts about the distance to the axis, and applies paraxial Snell's law and the law of reflection [43] to the orientation of the ray as shown in Fig. 2.

Finally, we can recursively apply these predicates to define the behavior of a ray going through a series of optical components in an arbitrary optical system, given as follows:

Definition 8 *(Valid ray behavior in an optical system).*

```
⊢def ∀ sr1 sr2 h h' fs cs rs i y0 θ0 y1 θ1 y2 θ2 y3 θ3 n d n' d'.
  C1: (is_valid_ray_in_system (sr1,sr2,[]) (CONS h cs,fs) ⇔ F) ∧
  C2: (is_valid_ray_in_system (sr1,sr2,CONS h' rs) ([],fs) ⇔F) ∧
  C3: (is_valid_ray_in_system ((y0,θ0),(y1,θ1),[]) ([],n,d) ⇔
       is_valid_ray_in_free_space (y0,θ0) (y1,θ1) (n,d)) ∧
  C4: (is_valid_ray_in_system ((y0,θ0),(y1,θ1),
       CONS ((y2,θ2),y3,θ3) rs) (CONS ((n',d'),i,ik) cs,n,d) ⇔
      (is_valid_ray_in_free_space (y0,θ0) (y1,θ1) (n',d') ∧
       is_valid_ray_at_interface (y1,θ1) (y2,θ2) n'
       (head_index (cs,n,d)) i)) ∧
      (is_valid_ray_in_system ((y2,θ2),(y3,θ3),rs) (cs,n,d))
```

where the first two cases (C1 and C2) describe the two situations where the length of the ray and optical system are not the same. The case C3 describes the situation when the optical system only consists of a free space. The last case recursively ensures the valid behavior of ray at each interface of the optical system. The behavior of a ray going through a series of optical components is thus completely defined.

3.3. Ray-transfer matrices of optical components

The main strength of ray optics is its matrix formalism [43], which provides an efficient way to model all optical components in the form of a matrix. Indeed, a matrix relates the input and the output ray by a linear relation. For example, in case of free space, the input and output ray parameters are related by two linear equations, i.e., $y_1 = y_0 + d * \theta_0$ and $\theta_1 = \theta_0$, which further can be described as a matrix (also called ray-transfer matrix of free space). We prove the ray-transfer matrices of all optical interfaces (Definition 3). The availability of verified ray-transfer matrices in our framework is quite handy as it helps to reduce the interactive verification efforts for the applications. Our next goal is to formally prove that any optical interface can be described by a general ray-transfer-matrix relation. Mathematically, this relation is described in the following theorem:

Theorem 4 *(Ray-transfer-matrix any interface).*

```
⊢ ∀ n0 n1 y0 θ0  y1 θ1  i. is_valid_interface i ∧
  is_valid_ray_at_interface (y0,θ0) (y1,θ1) n0 n1 i ∧
```

$$0 < n_0 \wedge 0 < n_1 \Rightarrow \begin{bmatrix} y_1 \\ \theta_1 \end{bmatrix} = \texttt{interface\_matrix}\ n_0\ n_1\ i\ {**} \begin{bmatrix} y_0 \\ \theta_0 \end{bmatrix}$$

where `interface_matrix` *accepts the refractive indices (*n_0 *and* n_1*) and interface (*i*), and returns corresponding matrix of the system. In the above theorem, both assumptions ensure the validity of the interface (*i*) and behavior of ray at each interface, respectively. We prove this theorem using the case splitting on interface* i*.*

We can trace the input ray R_i through an optical system consisting of n optical components by the composition of ray-transfer matrices of each optical component as described in Equation (1). It is important to note that in this equation, individual matrices of optical components are composed in a reverse order. We formalize this fact with the following recursive definition:

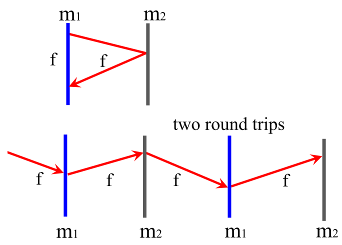

Fig. 3. Ray behavior inside an optical resonator.

Definition 9 *(System composition).*

```
⊢def system_composition ([],fs) = free_space_matrix fs ∧
     system_composition (CONS ((n',d'),i) cs,n,d) =
     system_composition (cs,n,d) **
     interface_matrix n' (head_index (cs,n,d)) i **
     free_space_matrix (n',d')
```

where the type of `system_composition` is : `optical_system` → $\mathbb{R}^{2\text{x}2}$, i.e., it takes an optical systems and returns a (2 × 2) matrix. The function `system_composition` is defined by two cases, i.e., if an optical system consists of only free space, it returns the corresponding matrix and if an optical system consists of a list of optical components (`cs`), it returns the product of corresponding matrices in a reversed order. Here, (n′, d′) represents the second free space in the system.

Finally, we verify the generalized ray-transfer-matrix relation for an arbitrary optical system which is valid for any optical and ray. We verify this relation in the following theorem:

Theorem 5 *(Ray-transfer-matrix for optical system).*

```
⊢ ∀ sys ray. is_valid_optical_system sys ∧
  is_valid_ray_in_system ray sys ⇒
  let (y0,θ0),(y1,θ1),rs = ray in
  let yn,θn = last_ray_at_point ray in
```

$$\begin{bmatrix} y_n \\ \theta_n \end{bmatrix} = \texttt{system\_composition sys} \ast\ast \begin{bmatrix} y_0 \\ \theta_0 \end{bmatrix}$$

where the parameters `sys` and `ray` represent the optical system and the ray, respectively. The function `last_ray_at_point` returns the last `ray_at_point` of the ray in the system. Both assumptions in the above theorem ensure the validity of the optical system and the good behavior of the ray in the system. We prove this theorem using induction on the length of the system and by using previous results and definitions.

4. Formalization of periodic optical systems

The analysis of periodic optical systems (or resonators) involves the study of infinite rays, or, equivalently, an infinite set of finite rays. Indeed, a resonator is a closed structure terminated by two reflected interfaces and a ray reflects back and forth between these interfaces. For example, consider a simple plane-mirror resonator as shown in Fig. 3. Let m_1 be the first mirror, m_2 the second one, and f the free space in between. Then the analysis involves the study of the ray as it goes through f, then reflects on m_2, then travels back through f, then reflects again on m_1, and starts over. So we have to consider the ray going through the "infinite" path $f, m_2, f, m_1, f, m_2, f, m_1, \ldots$, or, using regular expressions notations, $(f, m_2, f, m_1)^*$. The main purpose of stability analysis is to ensure that this infinite ray remains inside the cavity. This is equivalent to consider that, for every n, the ray going through the path $(f, m_2, f, m_1)^n$ remains inside the cavity. This allows to reduce the study of an infinite path to an infinite set of finite paths. On the other hand, in case of chaos generation, the main idea is to reproduce a particular pattern infinitely many times.

Our formalization (which is inspired by the optics literature) fixes the path of any considered ray. Since we want to consider an infinite set of finite-path rays, we should thus consider an infinite set of optical systems. This has been naturally achieved by optics engineers by "unfolding" the resonator as many times as needed, depending on the considered ray. For instance, consider again the above example of a plane-mirror resonator: if we want to observe a ray going back and forth only once through the cavity, then we should consider the optical system made of f, m_1, f, m_2; however, if we want to study the behavior of rays which make two round-trips through the cavity, then we consider a *new* optical system $f, m_1, f, m_2, f, m_1, f, m_2$ as shown in Fig. 3; and similarly for more round-trips.

In our formalization, we want the user to provide only the minimum information so that HOL Light generates automatically the unfolded systems. Therefore, we do not define resonators as just optical systems but we define a dedicated type for them. In their most usual form, resonators are made of two reflecting interfaces and a list of components in between. We thus define the following type:

```
resonator = interface # optical_component list # free_space # interface
```

Note that the additional free space in the type definition is required because the `optical_component` type only contains one free space (the one before the interface, not the one after).

Example. We can model the two mirror resonator (i.e., two plane mirrors and free space `fs`) of Fig. 3 as follows:

```
⊢∀ fs. two_mirror_res fs = (plane, [],fs, plane):resonator
```

We formally prove that a variable of type resonator can be decomposed into its constituents, i.e, interfaces, free space and a list of optical components:

Theorem 6 *(Optical resonator decomposition).*

```
⊢ ∀P. (∀res. P res) ⇔ (∀i1 cs fs i2. P (i1,cs,fs,i2))
```

Similar to the ray optics formalization, we introduce a predicate to ensure that a value of type `resonator` indeed models a real resonator:

Definition 10 *(Valid optical resonator).*

```
⊢def ∀i₁ cs fs i₂.
  is_valid_resonator ((i₁,cs,fs,i₂):resonator)⇔
   is_valid_interface i₁ ∧ ALL is_valid_optical_component cs ∧
   is_valid_free_space fs  ∧ is_valid_interface i₁
```

In our formalization, we develop a tactic `VALID_RESONATOR_TAC` which can automatically verify the validity of an optical resonator. We now present the formalization about the unfolding of a resonator as mentioned above. The first step in this process is to define a function `round_trip` which returns the list of components corresponding to one round-trip in the resonator:

Definition 11 *(Round trip).*

```
⊢def ∀i2 i1 cs fs.
    round_trip (i1,cs,fs,i2) =
    APPEND cs (CONS (fs,i2)
    let cs',fs1 = optical_components_shift cs fs in
        MAP (λa. sign_cor_interface a)
            (REVERSE (CONS (fs1,i1) cs'))))
```

where `APPEND` is a HOL Light library function which appends two lists and `REVERSE` reverses the order of elements of a list. The function `optical_component_shift cs fs` shifts the free spaces of `cs` from right to left, introducing `fs` to the right; the leftmost free space which is "ejected" is also returned by the function. This manipulation is required because unfolding the resonator entails the reversal of the components for the return trip. The function `sign_cor_interface` takes care of the correct sign of radius of curvature of spherical interfaces, i.e., `R` of convex and `-R` for concave interface. Similarly, we can define the notion of half round trip which is important in the study of chaotic optical resonators.

Definition 12 *(Half round trip).*

```
⊢def ∀fs2 i1 cs fs1 i2.
    half_round_trip (i1,cs,fs1,i2) fs2 =
    APPEND (APPEND [fs2,i1] cs) [fs1,i2],1,0
```

We can now define the unfolding of a resonator as follows:

Definition 13 *(Unfold resonator).*

```
⊢def ∀i1 cs fs i2.
    unfold_resonator ((i1,cs,fs,i2):resonator) N =
    list_pow (round_trip (i1,cs,fs,i2)) N,(head_index (cs,fs),0)
```

where `list_pow L n` concatenates `n` copies of the list `L`. The argument `N` represents the number of times we want to unfold the resonator. Note that the output type is `optical_system`, therefore all the functions and theorems of Section 3 can be used for an unfolded resonator.

We verify a key property which states that `optical_components_shift` always produces a valid structure of a given optical resonator if the list of components and free space are valid. The formal statement of this property is given in the following theorem:

Theorem 7 *(Valid optical component shift).*

```
⊢ ∀cs fs.
    ALL is_valid_optical_component cs ∧
    is_valid_free_space fs
    ⇒ (let cs',fs' = optical_components_shift cs fs in
        ALL is_valid_optical_component cs' ∧ is_valid_free_space fs')
```

In Section 3, we described the functions `head_index` and `system_composition` which provides the refractive index of the next optical element and composition of the matrices of optical components, respectively. Here, we provide two properties of these functions which are important to reason about optical resonators:

Theorem 8 *(Head index for round trip).*

```
⊢ ∀i1 cs fs i2.
    head_index (round_trip (i1,cs,fs,i2),fs) =
    head_index (cs,fs)
```

Theorem 9 *(System composition append).*

```
⊢ ∀cs1 cs2 fs.
    system_composition (APPEND cs1 cs2,fs) =
    system_composition (cs2,fs) **
    system_composition (cs1,head_index (cs2,fs),0)
```

Theorem 8 states that retrieving `head_index` does not depend on the two reflecting interfaces of a resonator whereas Theorem 9 describes the application of `system_composition` if the system is made of two appended component lists.

It is important to note that `unfold_resonator` provides the unfolded resonator structure which has the same type as of an optical system. Since an optical systems can be described by a matrix, unfolding a resonator is equivalent to multiplying that matrix n-times. We prove this fact in the following theorem:

Theorem 10 *(Unfold resonator matrix).*

```
⊢ ∀n res.
    system_composition (unfold_resonator res n) =
    system_composition (unfold_resonator res 1) pow n
```

We mainly prove this theorem using the induction on n along with some other already proved theorems (e.g., Theorem 6).

This concludes our formalization of optical resonators. In summary, we formalized the basic notions for optical resonators which included the new type definition and corresponding validity constraints and helper functions such as round trip and unfolding of an optical resonator. The notable feature of our formalization is its generic nature, as we can model optical resonators with any number of optical components composed by the basic types of interfaces formalized in Section 3. In the next sections, we present the formalization of resonator stability and chaotic maps.

5. Formalization of stability and chaos in HOL

In this section, we present the formal verification of the stability conditions and chaotic map generation inside arbitrary optical resonators.

5.1. Formalization of stability

The stability of a resonator depends on the properties and arrangement of its components, e.g., curvature of mirrors or lenses, and distance between them. In order to determine whether a given optical resonator is stable, we need to analyze the ray behavior after many round trips by "unfolding" N times the resonator description, and compute the corresponding ray-transfer matrix. This is equivalent to taking the ray-transfer matrix corresponding to one round-trip and then raising it to the Nth power. For an optical resonator to be stable, the distance of the ray from the optical axis and its orientation should remain bounded whatever the value of N. This is formalized as follows:

Definition 14 *(Resonator stability).*

```
⊢def ∀ res.is_stable_resonator res ⇔
  (∀(r:ray). ∃ y θ. ∀ N. is_valid_ray_in_system r (unfold_resonator res N) ⟹
  (let yn,θn = last_single_ray r in abs(yn) ≤ y ∧ abs(θn) < θ))
```

where `res` and `abs` represent an optical resonator and absolute value of a real number, respectively. Note that in our definition of stability, a ray is not explicitly provided which implies that a resonator has be to stable for any injected ray.

For an arbitrary optical resonator, proving that a resonator satisfies the abstract condition of Definition 14 does not seem trivial at first. However, if the determinant of a resonator matrix M is 1 (which is the case in practice), optics engineers have known for a long time that having $-1 < \frac{M_{11}+M_{22}}{2} < 1$ is sufficient to ensure that the stability condition holds. The obvious advantage of this criterion is that it is immediate to check. This can actually be proved by using Sylvester's theorem [42], which states that for a matrix $M = [ABCD]$ such that $|M| = 1$ and $-1 < \frac{A+D}{2} < 1$, the following holds:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^N = \frac{1}{\sin(\theta)} \begin{bmatrix} A\sin[N\theta] - \sin[(N-1)\theta] & B\sin[N\theta] \\ C\sin[N\theta] & D\sin[N\theta] - \sin[(N-1)\theta] \end{bmatrix}$$

where $\theta = cos^{-1}[\frac{A+D}{2}]$. This theorem allows to prove that stability holds under the considered assumptions: indeed, N only occurs under a sine in the resulting matrix; since the sine itself is comprised between -1 and 1, it follows that the components of the matrix are obviously bounded, hence the stability. Formally, Sylvester's Theorem can be written in HOL Light as follows:

Theorem 11 *(Sylvester's theorem).*

$$\vdash \forall\ \mathtt{N\ A\ B\ C\ D}.\ \begin{vmatrix} \mathtt{A} & \mathtt{B} \\ \mathtt{C} & \mathtt{D} \end{vmatrix} = 1 \land -1 < \frac{\mathtt{A+D}}{2} \land \frac{\mathtt{A+D}}{2} < 1 \Rightarrow \mathtt{let}\ \theta = \mathtt{acs}(\frac{\mathtt{A+D}}{2})\ \mathtt{in}$$
$$\begin{bmatrix} \mathtt{A} & \mathtt{B} \\ \mathtt{C} & \mathtt{D} \end{bmatrix}^{\mathtt{N}} = \frac{1}{\sin(\theta)} \begin{bmatrix} \mathtt{A} * \sin[\mathtt{N}*\theta] - \sin[(\mathtt{N}-1)*\theta] & \mathtt{B}*\sin[\mathtt{N}*\theta] \\ \mathtt{C}*\sin[\mathtt{N}*\theta] & \mathtt{D}*\sin[\mathtt{N}*\theta] - \sin[(\mathtt{N}-1)*\theta] \end{bmatrix}$$

We prove Theorem 11 by induction on N and using the fundamental properties of trigonometric functions, matrices and determinants, and the automated decision procedures (`REAL_ARITH` and `REAL_FIELD`) available in HOL Light.

This allows to derive now the generalized stability theorem for a resonator as follows:

Theorem 12 *(Stability theorem).*

```
⊢ ∀ res. is_valid_resonator res ∧
  (∀ N. let M = system_composition (unfold_resonator res 1) in
 det M = 1 ∧ -1 < (M1,1+M2,2)/2 ∧ (M1,1+M2,2)/2 < 1) ⇒
  is_stable_resonator res
```

where $\mathtt{M}_{\mathtt{i,j}}$ *represents the element at column* `i` *and row* `j` *of the matrix.*

The formal verification of Theorem 12 requires the definition of stability (Definition 14) along with Theorem 11 (Sylvester's theorem) and Theorem 10. We also require to prove that an unfolded resonator remains structurally valid, as given in the following theorem:

Theorem 13 *(Valid unfold resonator).*

```
⊢ ∀ res. is_valid_resonator res ⇒
        (∀ n. is_valid_optical_system (unfold_resonator res n))
```

Note that our stability theorem (Theorem 12) is quite general and can be used to verify the stability of optical resonators which can be modeled using the laws of paraxial optics. In the next section, we present the formalization of chaotic maps and chaos generation inside an optical resonators.

5.2. Formalization of chaos in optical resonators

Chaos is a special behavior which is usually observed in dynamical systems where the output response posses a sensitive behavior for minor changes in the initial conditions or system parameters. A chaotic map is a dynamic function that exhibits chaotic behavior. Generally, chaotic maps can be discrete-time or continuous-time. In recent times, the chaotic behavior has been studied in almost all fields of science and engineering, e.g., electrical circuits, chemical, biological and mechanical systems. In optics, the phenomena of chaos is concerned with the dynamic nature of light. For example, chaos can be found in the output of a laser diode and the fluctuations of light inside an optical cavity or resonator [36]. Even though chaotic systems are unpredictable but they can be used for many important performance improvements, e.g., chaos in optical systems has been used for secure and high-speed transmission of messages in optical-fiber networks [6], efficient light energy storage [36] and fast random number generation [45].

In this paper, our main focus is to formalize the notion of chaos in optical resonators. The main idea behind this is to find the conditions in terms of the parameters of the resonators so that the trapped light follows some chaotic map. For example, the Duffing Map and Tinkerbell Map [4] are two important two-dimensional discrete-time chaotic maps, given in the follows equations, respectively:

$$y_{n+1} = \theta_n \tag{2}$$

$$\theta_{n+1} = -\beta y_n + (\alpha - (\theta_n)^2)\theta_n \tag{3}$$

$$y_{n+1} = (y_0 + \alpha) * y_n + (-\theta_n + \beta) * \theta_n \tag{4}$$

$$\theta_{n+1} = (2 * \theta_n + \gamma) * y_n + \delta * \theta_n \tag{5}$$

Note that y_n and θ_n are the scalar state variables and α, β, γ, and δ represent the map parameters.

The half round trip of an optical resonator provides the ray path from one terminating optical interface to the other as formalized in Definition 12. Formally, an optical resonator is considered to be chaotic if ray follows a particular chaotic map after every half round trip. We formalize the notion of chaotic resonator as follows:

Definition 15 *(Chaotic resonator).*

```
⊢def ∀ res map fs.  chaotic_resonator res map fs ⇔
   (∀ ray. is_valid_ray_in_system ray  (half_round_trip  res  fs) ⇒
   (let y0,θ0  =  fst_single_ray ray
    and yn,θn  =  last_single_ray ray in map (y0,θ0)  (yn,θn)))
```

where `resonator`, `map` and `fs` represent an optical resonator (`:resonator`), a chaotic map (`:A → bool`) and a free space, respectively.

Our definition of chaotic resonator is general and can be used to model any kind of optical resonator and any type of corresponding two-dimensional chaotic map. We formalize the Duffing Map in HOL Light as follows:

Definition 16 *(Duffing map).*

```
⊢def ∀ y0  θ0  y1  θ1  α  β.
  duffing_map  (y0,θ0)  (y1,θ1)  α  β  ⇔  y1  =  θ0  ∧  θ1  =  -β  *  y0  +  (α  -  θ0  *  θ0)  *  θ0
```

where `duffing_map` represents Equations (2)–(3).

We next formally derive the Duffing Map generation conditions for an arbitrary optical resonator. We start by proving that a Duffing Map can be represented in a matrix–vector form as follows:

Theorem 14 *(Duffing map matrix form).*

$$\vdash \forall\ \mathtt{y}_0\ \theta_0\ \mathtt{y}_1\ \theta_1\ \alpha\ \beta.\quad \mathtt{duffing\_map}\ (\mathtt{y}_0,\theta_0)\ (\mathtt{y}_1,\theta_1)\ \alpha\ \beta \Leftrightarrow \begin{bmatrix}\mathtt{y}_1\\ \theta_1\end{bmatrix} = \mathtt{duffing\_map\_matrix}\ \alpha\ \beta\ \theta_0\ {**} \begin{bmatrix}\mathtt{y}_0\\ \theta_0\end{bmatrix}$$

Where duffing_map_matrix is defined as follows:

Definition 17 *(Duffing matrix).*

$$\vdash_{def} \forall\ \alpha\ \beta\ \theta_0.\quad \mathtt{duffing\_map\_matrix}\ \alpha\ \beta\ \theta_0 = \begin{bmatrix}0 & 1\\ -\beta & (\alpha - \theta_0 * \theta_0)\end{bmatrix}$$

Finally, we can prove the chaos generation inside an optical resonator in the following theorem:

Theorem 15 *(Duffing map conditions).*

```
⊢ ∀ res fs α β.
  (∀ θ0. system_composition (re_hal_roun_trip res fs) = duffing_map_matrix α β θ0) ∧
     is_valid_free_space fs ∧ is_valid_resonator res ⇒
     chaotic_resonator res (λ(y0,θ0) (y1,θ1). duffing_map (y0,θ0) (y1,θ1) α β) fs
```

where the first assumption ensures that the ray-transfer matrix of the half round trip should be equivalent to the matrix of Duffing Map, i.e., the ray after each half round trip should follow the duffing map. The second and third assumptions ensure the valid architecture of a given resonator (res).

We mainly prove this theorem using Theorem 5 which states that any optical system can be described by a matrix and the following theorem, stating the validity of a half round trip of a resonator:

Theorem 16 *(Valid half round trip).*

```
⊢ ∀ fs res.
    is_valid_free_space fs ∧
    is_valid_resonator res ⇒
    is_valid_optical_system (half_round_trip res fs)
```

We conclude here the formalization of stability and chaos generation in optical resonators. Note that the conditions derived in Theorems 12 and 15 are quite general and can be utilized for arbitrarily complex optical resonators (which can be modeled using paraxial optics) as described in the sequel.

6. Applications

In this section, we consider two real-world optical resonators namely a Fabry Pérot resonator with a fiber rod lens and a ring resonator. We utilize our formalization of optical resonators, stability and chaos in optical resonators to formally verify the stability conditions for the Fabry Pérot resonator and chaotic map generation conditions for the ring resonator.

6.1. Formal stability analysis of Fabry Pérot resonator

The Fabry–Pérot (FP) resonator has numerous variants in terms of structure due to its wide scope of applications (e.g., wavelength division multiplexing [34], measurement of the refractive index of cancer cells [40], optical bio-sensing devices [8], etc.). Recently, a state-of-the-art FP core architecture has been proposed which overcomes the limitations of existing FP resonators [25]. In the new design, cylindrical mirrors are combined with a fiber rod lens (FRL) inside the cavity, to focus the beam of light in both transverse planes as shown in Fig. 4 (a). The fiber rod lens is used as light pipe which allows the transmission of light from one end to the other with relatively small leakage. Building a stable FP resonator requires the geometric constraints to be determined in terms of the radius of curvature of mirrors R and the free space propagation distance (d_{free_space}) using the stability analysis.

Note that the design shown in Fig. 4(a), has a 3-dimensional structure. We can still apply the ray-transfer-matrix approach to analyze the stability by dividing the given architecture into two planes, i.e., XZ and YZ planes. Now, the stability problem becomes a couple of planar problems which are still valid since the ray focusing behaviors in both directions (XZ and YZ) are decoupled. This is merely a consequence of the decomposition of Euclidean space vectors into its basis. This can be seen in Figs. 4(b) and 4(c), where the resonator is divided into two cross-sections. In the following, we present the stability verification in the YZ plane.

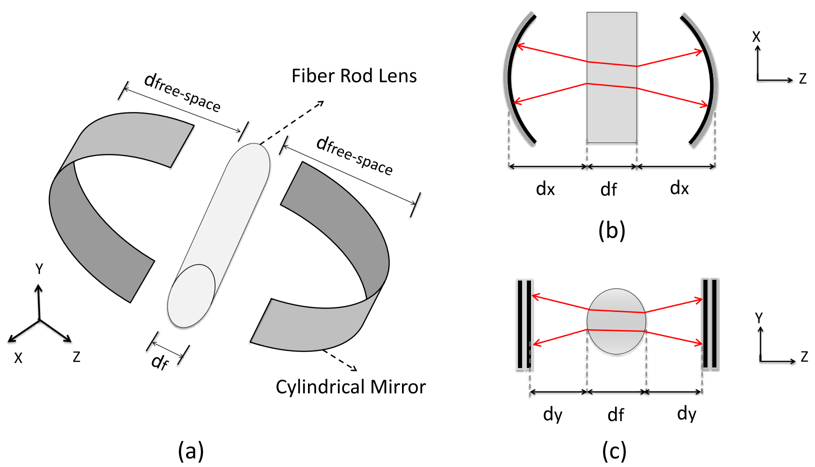

Fig. 4. Fabry Pérot (FP) resonator with fiber rod lens (a) 3-dimensional resonator design (b) cross-section view in the XZ plane (c) cross-section view in the YZ plane.

6.1.1. Stability constraints in YZ-plane

In the YZ cross-section (Fig. 4(c)), the curved mirrors become straight mirrors and the fiber rod lens acts as a converging lens. In this case, a ray that makes a round-trip in the cavity follows (from left to right) this sequence:

- Propagation through free space of length d_y and refractive index 1.
- Refraction through the curved interface with radius of curvature $\frac{d_f}{2}$.
- Propagation through a free space of length d_y.
- Refraction through the curved interface with radius of curvature $-\frac{d_f}{2}$.
- Propagation through free space of length d_y and refractive index 1.
- Reflection from the plane interface.

We formally model this system description as follows:

Definition 18 *(FP-FRL resonator in YZ-plane).*

```
⊢def ∀ dy nf df.
 fp_frl_resonator_yz dy nf df =
 plane_reflected,[(1,dy),spherical_transmitted(df/2);(nf,df),
 spherical_transmitted(-df/2)], (1,dy), plane_reflected
```

where the function `fp_frl_resonator_yz` takes as parameters the free space of length (`dy`), the length of fiber rod lens (`df`) and the refractive index (`nf`) and returns an optical resonator. Now, we follow three steps to formally derive the stability constraints.

Step 1: We check the validity of the model `fp_frl_resonator_yz` under the sufficient conditions as given below:

Theorem 17 *(Validity of FP-FRL in YZ-plane).*

```
⊢ ∀ dy nf df.
    0 < dy ∧ 0 < df ∧ 0 < nf ⇒
    is_valid_resonator (fp_frl_resonator_yz dy nf df)
```

Step 2: We verify the equivalent matrix expression of the FP resonator in the YZ plane as follows:

Theorem 18 *(Matrix for FP-FRL in YZ-plane).*

```
⊢ ∀ dy df nf.  0 < dy ∧  0 < df ∧ 0 < nf ⇒
    system_composition (fp_frl_resonator_yz dy df nf) =
```

$$\begin{bmatrix} -\frac{df(-2+nf)+4dy(-1+nf))}{df\cdot nf} & \frac{(df+2dy)(df-2dy(-1+nf))}{df\cdot nf} \\ \frac{4-4nf}{df\cdot nf} & -\frac{df(-2+nf)+4dy(-1+nf)}{df\cdot nf} \end{bmatrix}$$

Table 1
Stability ranges for FP resonator.

R (μm)	d (μm)	Stable in XZ plane	Stable in YZ plane
140	$133 < d$	NO	NO
140	(27.5, 35)	YES	NO
140	(97.5, 132.9)	NO	YES
140	(38, 97)	YES	YES

Step 3: Finally, we formally verify the stability of the FP resonator in the YZ plane as follows:

Theorem 19 *(Stability in YZ-plane).*

$$\vdash \forall\, dy\ df\ nf.\ \ 0 < dy \wedge\ 0 < df \wedge 0 < nf$$
$$0 < 1 - \frac{2}{nf} + (4\frac{dy}{df})(1 - \frac{1}{nf}) \wedge$$
$$1 - \frac{2}{nf} + (4\frac{dy}{df})(1 - \frac{1}{nf}) < 1 \Rightarrow \texttt{is\_stable\_resonator (fp\_frl\_resonator\_yz dy df nf)}$$

The verification of this theorem is a direct consequence of the generalized stability theorem (verified in the previous section) and Steps 1 and 2. Similarly, we can derive the stability conditions in the XZ plane.

It is important to note that for the case of the FP resonator with fiber rod lens, we have obtained two sets of stability constraints, i.e., one in the XZ plane and another in the YZ plane (Theorem 19). In fact, the resonator can be stable in one plane and unstable in the other. Therefore, stability constraints in both planes have to be satisfied. In real-world scenarios, the most fundamental step is to find the allowable values of the parameters associated with the resonator such as radius of curvature and the width of free space. The verification of above theorems has been done in a generic form, i.e., we derive the stability constraints for arbitrary values of R, d_x, d_f and n_f.

6.1.2. Automated tactic for stability ranges

We further demonstrate the strength of our approach by the verification of stability constraints used as the guidelines for the fabrication of FP resonators, reported in [26]. In the design, it is considered that $d_x = d_y = d$ and the values of n_f and d_f are fixed and equal to 1.47 and 125 μm, respectively. The main goal is to find the ranges of d where the resonator is stable in both planes. We developed a tactic (`STABILITY_PROVE_TAC`) which can automatically verify that the resonator is stable under the given range of parameters. For example, one particular case is given as follows:

Input:

```
STABILITY_PROVE_TAC `(d IN real_interval (#27.5 * #0.000001,#35 * #0.000001) ==>
                      is_stable_resonator (FP_XZ_RES d))`;;
```

Output:

```
CPU time (user): 4.878
val it : thm = |- d IN real_interval(#27.5 * #0.000001,#35 * #0.000001) ==>
                 is_stable_resonator (FP_XZ_RES d)
```

where `real_interval` represents a closed interval `(a,b)`. The output shows that the stability conditions are verified for the given resonator configuration. Note that the CPU time is represented in seconds.

Table 1 provides the typical dimensional ranges, corresponding to different practical situations: in the first case, stability is not reached at all, in the second case, we have stability along the X axis and instability along the Y axis, in the third case, we have instability along the X axis and stability along the Y axis. In the fourth case, we fulfill the stability conditions along both X and Y axis.

6.2. Chaos generation conditions for ring resonators

In the last few decades, optical phase conjugation (OPC) [4] has been widely studied in lasers and nonlinear optics. Physically, the OPC describes the relation among light beams propagating in opposite directions with reversed wave front and identical transverse amplitude distributions. The main applications of phase conjugation are high-brightness laser oscillator/amplifier systems, laser target-aiming systems, long distance optical fiber communications with ultra-high bit-rate, etc. In this paper, we consider a ring resonator OPC which is mainly based on the two-dimensional chaotic maps. This is usually done by placing an intracavity element which is responsible of generating a particular chaotic map whose state is determined by its previous state. Our main intent is to formally show that the introduction of a specific element within a ring-phase-conjugated resonator can produce a Duffing Map [4] inside the resonator. The architecture of the ring-phase

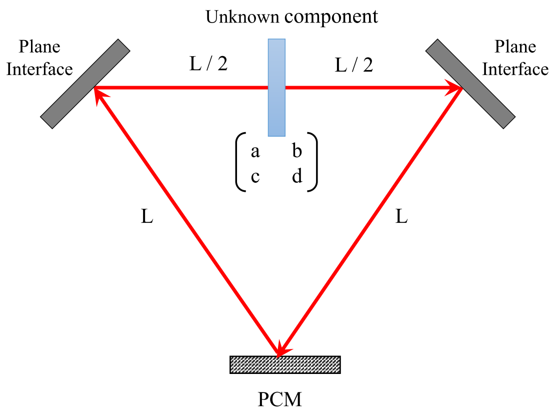

Fig. 5. Phase conjugated ring resonator.

conjugated resonator consisting of two plane mirrors, a phase conjugate mirror along with an unknown optical element is shown in Fig. 5.

Our datatype for optical interfaces is general and we can model an unknown element by a matrix. The half round-trip of a ray is based on the following sequence:

- Propagation through free space of length $\frac{L}{2}$ and refractive index n.
- Propagation through an unknown element which has parameters, a, b, c and d.
- Propagation through free space of length $\frac{L}{2}$ and refractive index n.
- Reflection from plane mirror.
- Propagation through free space of length L and refractive index 1.
- Reflection from phase conjugated mirror (PCM).
- Propagation through free space of length L and refractive index 1.

We formally define the structure of the phase-conjugated ring resonator as follows:

Definition 19 *(Phase conjugated ring resonator).*

```
⊢def ∀a b c d n L. ring_resonator a b c d L n =
  plane_reflected, [(n,L/2),unknown a b c d; (n,L/2),plane_reflected],(n,L), pcm
```

where a, b, c and d represent parameters of the unknown component whereas n and L represent the refractive index and length of the free space, respectively.

We first verify the validity of the ring resonator as follows:

Theorem 20 *(Valid ring resonator).*

```
⊢ ∀ n L. 0 < n ∧ 0 ≤ L ⇒ is_valid_resonator (ring_resonator a b c d L n)
```

At this point, we have already developed the formal model of the ring resonator along with the verification of its structural constraints. Our ultimate goal in the analysis of this resonator is to formally derive the conditions on a, b, c and d, so that the rays inside the resonator follows the duffing map. This leads to the following theorem:

Theorem 21 *(Chaos verification in ring resonator).*

$$\vdash \forall\ n\ L\ \alpha\ \beta.$$
$$(\forall\ \theta_0.\ a = -\frac{(3 * \beta * L)}{2}\ \wedge$$
$$b = \frac{1}{4} * (4 + 6 * \alpha * L + 9 * \beta * L * L - 6 * L * \theta_0 * \theta_0)\ \wedge$$
$$c = \beta \wedge d = -\alpha - \frac{3}{2} * L * \beta + \theta_0 * \theta_0) \Rightarrow$$
$$\text{chaotic\_resonator (ring\_resonator a b c d L n) } (\lambda(y_0,\theta_0)\ (y_1,\theta_1).$$
$$\text{duffing\_map } (y_0,\theta_0)\ (y_1,\theta_1)\ \alpha\ \beta)\ (n,L)$$

The proof of this theorem is mainly based on Theorems 15 and 20.

In this application, we formally prove that the introduction of a particular map generating device in a ring optical phase-conjugated resonator can generate a ray with the behavior of a specific two-dimensional chaotic map (e.g., Duffing Map). In particular, we explicitly derive the conditions on the unknown component [a,b,c,d] which are necessary to produce the Duffing Map inside the resonator. The procedure described in this section can be used to derive similar conditions for other chaotic maps such as the Tinkerbell Map.

7. Discussion

In this paper, we proposed to use the HOL Light proof assistant for the verification of some important behavioral properties of periodic optical systems. The main idea was to leverage the modular nature of optical systems and develop formal models in a hierarchical style. Indeed, the use of HOL light allowed us to verify generic properties which can be reused and applied for different applications such as the FP resonator, the ring resonator, etc. It is important to note that all theorems presented in our work are verified under universal quantification of system parameters (e.g., radius of curvature and width of free space) unlike other numerical approaches (e.g., reZonator [32], a numerical analysis software for resonators) where the results hold only for specific values of these parameters. The main benefit of formal proofs is that all the underlying assumptions can be seen explicitly and the proof-steps can be verified mechanically using a theorem prover, which otherwise is not possible for paper-and-pencil based proofs. In spite of the fact that our approach required a significant amount of time to formalize the underlying theories of optics, we believe that our formal development can be used to verify some critical properties of periodic optical systems (e.g., resonator stability) which require infinite set of rays. Lastly, it is worth mentioning that the formal stability analysis of the FP resonator with fiber rod lens allowed us to find some discrepancy in the paper-and-pencil based proof approach presented in [27]. Particularly, the order of matrix multiplication in Equations (16) and (24) in [27] should be reversed, so as to obtain the correct stability constraints. This is one of the main strengths of theorem proving where the soundness is assured for every step during the proof of system properties.

The formal verification of periodic optical systems indicates some important points: theorem proving systems have reached to the maturity, where complex physical models can be expressed with less efforts than ever before; and formal methods can assist in the verification of futuristic optical systems which are largely becoming part of critical applications such as military setups, biomedical surgeries and space missions. However, the utilization of higher-order-logic theorem proving in an industrial settings (particularly, physical systems) is very challenging due to the large amount of time required to formalize the underlying theories in HOL. We believe that an important factor is the gap between the theorem proving and engineering communities which limits its usage in industrial settings. One of the several solutions to tackle this issue is the continuous formal development of optics theories including libraries of most frequently used optical components and devices which can ultimately reduce the cost of using formal methods (particularly theorem proving) as an integral part of the physical systems design and verification cycle. The work presented in this paper can be considered as a one step towards this goal with more efforts to follow in the same or closely related disciplines such as quantum optics, photonic signal processing and optoelectronics.

8. Conclusion

This paper reported a formal approach to formally verify the properties of periodic optical systems which are widely used in practical applications ranging from biomedical devices to aerospace equipments. In particular, we discussed the hierarchical modeling scheme for optical systems and their corresponding implementation in the HOL Light proof assistant. We demonstrated the effectiveness of our approach by verifying the stability and chaos generating conditions for a couple of real-world optical resonators, i.e., FP resonator and ring resonator. We believe that our work is not only related to applying formal methods in optical systems but also introducing a systems perspective in optics. In the optical engineering literature, many systems other than geometrical optics can also be modeled based on the transfer-matrix approach. Some examples of such systems are laser systems [39], frequency division multiplexing/demultiplexing [11] and photonic signal processing applications [48]. The formalization described in this paper can be used as a guideline for the formal analysis of above mentioned systems. It may require some modifications about new datatypes for underlying components and corresponding physical behavior, though.

Lastly, we believe that our reported work complements traditional techniques (e.g., paper-and-pencil based proofs and numerical simulations) and can be used to verify the correctness of safety-critical optical systems. As mentioned by Avigad and Harrison [7], the availability of formalized libraries of mathematics can attract mathematicians to use interactive theorem proving for verifying key lemmas in their work, so is the case of optical engineers. A potential utilization of our work is to formally verify the correctness and soundness of ray tracing algorithms [43], which are included in almost all optical systems design tools.

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