# 1 CAS Theory

**Built:** 03 July 2019
**Parent Theories:** DFT_Gates_def_PROB

## 1.1 Definitions

[indep_vars_10_def]

$$\vdash \forall p\ M\ A_0\ A_1\ A_2\ A_3\ A_4\ A_5\ A_6\ A_7\ A_8\ A_9.\ \texttt{indep\_vars\_10}\ p\ M\ A_0\ A_1\ A_2\ A_3\ A_4\ A_5\ A_6\ A_7\ A_8\ A_9 \iff \texttt{indep\_vars}\ p\ (\lambda i.\ M)\ (\lambda i.\ \textbf{if}\ i = 0\ \textbf{then}\ A_0\ \textbf{else if}\ i = 1\ \textbf{then}\ A_1\ \textbf{else if}\ i = 2\ \textbf{then}\ A_2\ \textbf{else if}\ i = 3\ \textbf{then}\ A_3\ \textbf{else if}\ i = 4\ \textbf{then}\ A_4\ \textbf{else if}\ i = 5\ \textbf{then}\ A_5\ \textbf{else if}\ i = 6\ \textbf{then}\ A_6\ \textbf{else if}\ i = 7\ \textbf{then}\ A_7\ \textbf{else if}\ i = 8\ \textbf{then}\ A_8\ \textbf{else}\ A_9)\ \{0;\ 1;\ 2;\ 3;\ 4;\ 5;\ 6;\ 7;\ 8;\ 9\}$$

[UNIONL_def]

$$\vdash (\texttt{UNIONL}\ [\,] = \{\,\}) \land \forall s\ ss.\ \texttt{UNIONL}\ (s::ss) = s \cup \texttt{UNIONL}\ ss$$

## 1.2 Theorems

[add_assoc_10_var]

$$\vdash \forall A_1\ A_2\ A_3\ A_4\ A_5\ A_6\ A_7\ A_8\ A_9\ A_{10}.\ A_1 \neq \texttt{PosInf} \land A_2 \neq \texttt{PosInf} \land A_3 \neq \texttt{PosInf} \land A_4 \neq \texttt{PosInf} \land A_5 \neq \texttt{PosInf} \land A_6 \neq \texttt{PosInf} \land A_7 \neq \texttt{PosInf} \land A_8 \neq \texttt{PosInf} \land A_9 \neq \texttt{PosInf} \land A_{10} \neq \texttt{PosInf} \Rightarrow (A_1 + (A_2 + (A_3 + (A_4 + (A_5 + (A_6 + (A_7 + (A_8 + (A_9 + A_{10})))))))) = A_1 + A_2 + A_3 + A_4 + A_5 + A_6 + A_7 + A_8 + A_9 + A_{10})$$

[add_assoc_20_var]

$$\vdash \forall A_1\ A_2\ A_3\ A_4\ A_5\ A_6\ A_7\ A_8\ A_9\ A_{10}\ A_{11}\ A_{12}\ A_{13}\ A_{14}\ A_{15}\ A_{16}\ A_{17}\ A_{18}\ A_{19}\ A_{20}.\ A_1 \neq \texttt{PosInf} \land A_2 \neq \texttt{PosInf} \land A_3 \neq \texttt{PosInf} \land A_4 \neq \texttt{PosInf} \land A_5 \neq \texttt{PosInf} \land A_6 \neq \texttt{PosInf} \land A_7 \neq \texttt{PosInf} \land A_8 \neq \texttt{PosInf} \land A_9 \neq \texttt{PosInf} \land A_{10} \neq \texttt{PosInf} \land A_{11} \neq \texttt{PosInf} \land A_{12} \neq \texttt{PosInf} \land A_{13} \neq \texttt{PosInf} \land A_{14} \neq \texttt{PosInf} \land A_{15} \neq \texttt{PosInf} \land A_{16} \neq \texttt{PosInf} \land A_{17} \neq \texttt{PosInf} \land A_{18} \neq \texttt{PosInf} \land A_{19} \neq \texttt{PosInf} \land A_{20} \neq \texttt{PosInf} \Rightarrow$$

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$(A_{14}$ +
$(A_{15}$ + $(A_{16}$ + $(A_{17}$ + $(A_{18}$ + $(A_{19}$ + $A_{20}))))))))))))))))))$ =
$A_1$ + $A_2$ + $A_3$ + $A_4$ + $A_5$ + $A_6$ + $A_7$ + $A_8$ + $A_9$ + $A_{10}$ + $A_{11}$ +
$A_{12}$ + $A_{13}$ + $A_{14}$ + $A_{15}$ + $A_{16}$ + $A_{17}$ + $A_{18}$ + $A_{19}$ + $A_{20})$

[add_assoc_30_var]

$\vdash \forall A_1\ A_2\ A_3\ A_4\ A_5\ A_6\ A_7\ A_8\ A_9\ A_{10}\ A_{11}\ A_{12}\ A_{13}\ A_{14}\ A_{15}\ A_{16}\ A_{17}$
$A_{18}\ A_{19}\ A_{20}\ A_{21}\ A_{22}\ A_{23}\ A_{24}\ A_{25}\ A_{26}\ A_{27}\ A_{28}\ A_{29}\ A_{30}.$
$A_1 \neq$ PosInf $\wedge$ $A_2 \neq$ PosInf $\wedge$ $A_3 \neq$ PosInf $\wedge$ $A_4 \neq$ PosInf $\wedge$
$A_5 \neq$ PosInf $\wedge$ $A_6 \neq$ PosInf $\wedge$ $A_7 \neq$ PosInf $\wedge$ $A_8 \neq$ PosInf $\wedge$
$A_9 \neq$ PosInf $\wedge$ $A_{10} \neq$ PosInf $\wedge$ $A_{11} \neq$ PosInf $\wedge$
$A_{12} \neq$ PosInf $\wedge$ $A_{13} \neq$ PosInf $\wedge$ $A_{14} \neq$ PosInf $\wedge$
$A_{15} \neq$ PosInf $\wedge$ $A_{16} \neq$ PosInf $\wedge$ $A_{17} \neq$ PosInf $\wedge$
$A_{18} \neq$ PosInf $\wedge$ $A_{19} \neq$ PosInf $\wedge$ $A_{20} \neq$ PosInf $\wedge$
$A_{21} \neq$ PosInf $\wedge$ $A_{22} \neq$ PosInf $\wedge$ $A_{23} \neq$ PosInf $\wedge$
$A_{24} \neq$ PosInf $\wedge$ $A_{25} \neq$ PosInf $\wedge$ $A_{26} \neq$ PosInf $\wedge$
$A_{27} \neq$ PosInf $\wedge$ $A_{28} \neq$ PosInf $\wedge$ $A_{29} \neq$ PosInf $\wedge$ $A_{30} \neq$ PosInf $\Rightarrow$
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$(A_{25} +$
$(A_{26} +$
$(A_{27} + (A_{28} + (A_{29} + A_{30}))))))))))))))))))))))))))))) =$
$A_1 + A_2 + A_3 + A_4 + A_5 + A_6 + A_7 + A_8 + A_9 + A_{10} + A_{11} +$
$A_{12} + A_{13} + A_{14} + A_{15} + A_{16} + A_{17} + A_{18} + A_{19} + A_{20} +$
$A_{21} + A_{22} + A_{23} + A_{24} + A_{25} + A_{26} + A_{27} + A_{28} + A_{29} +$
$A_{30})$

[`add_assoc_40_var`]

$\vdash \forall A_1\ A_2\ A_3\ A_4\ A_5\ A_6\ A_7\ A_8\ A_9\ A_{10}\ A_{11}\ A_{12}\ A_{13}\ A_{14}\ A_{15}\ A_{16}\ A_{17}$
$A_{18}\ A_{19}\ A_{20}\ A_{21}\ A_{22}\ A_{23}\ A_{24}\ A_{25}\ A_{26}\ A_{27}\ A_{28}\ A_{29}\ A_{30}\ A_{31}$
$A_{32}\ A_{33}\ A_{34}\ A_{35}\ A_{36}\ A_{37}\ A_{38}\ A_{39}\ A_{40}.$
$A_1 \neq \texttt{PosInf} \land A_2 \neq \texttt{PosInf} \land A_3 \neq \texttt{PosInf} \land A_4 \neq \texttt{PosInf} \land$
$A_5 \neq \texttt{PosInf} \land A_6 \neq \texttt{PosInf} \land A_7 \neq \texttt{PosInf} \land A_8 \neq \texttt{PosInf} \land$
$A_9 \neq \texttt{PosInf} \land A_{10} \neq \texttt{PosInf} \land A_{11} \neq \texttt{PosInf} \land$
$A_{12} \neq \texttt{PosInf} \land A_{13} \neq \texttt{PosInf} \land A_{14} \neq \texttt{PosInf} \land$
$A_{15} \neq \texttt{PosInf} \land A_{16} \neq \texttt{PosInf} \land A_{17} \neq \texttt{PosInf} \land$
$A_{18} \neq \texttt{PosInf} \land A_{19} \neq \texttt{PosInf} \land A_{20} \neq \texttt{PosInf} \land$
$A_{21} \neq \texttt{PosInf} \land A_{22} \neq \texttt{PosInf} \land A_{23} \neq \texttt{PosInf} \land$
$A_{24} \neq \texttt{PosInf} \land A_{25} \neq \texttt{PosInf} \land A_{26} \neq \texttt{PosInf} \land$
$A_{27} \neq \texttt{PosInf} \land A_{28} \neq \texttt{PosInf} \land A_{29} \neq \texttt{PosInf} \land$
$A_{30} \neq \texttt{PosInf} \land A_{31} \neq \texttt{PosInf} \land A_{32} \neq \texttt{PosInf} \land$
$A_{33} \neq \texttt{PosInf} \land A_{34} \neq \texttt{PosInf} \land A_{35} \neq \texttt{PosInf} \land$
$A_{36} \neq \texttt{PosInf} \land A_{37} \neq \texttt{PosInf} \land A_{38} \neq \texttt{PosInf} \land$
$A_{39} \neq \texttt{PosInf} \land A_{40} \neq \texttt{PosInf} \Rightarrow$
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$(A_{33}$ +
$(A_{34}$ +
$(A_{35}$ +
$(A_{36}$ +
$(A_{37}$ +
$(A_{38} + (A_{39} + A_{40}))))))))))))))))))))))))))))))))$
$A_1 + A_2 + A_3 + A_4 + A_5 + A_6 + A_7 + A_8 + A_9 + A_{10} + A_{11} +$
$A_{12} + A_{13} + A_{14} + A_{15} + A_{16} + A_{17} + A_{18} + A_{19} + A_{20} +$
$A_{21} + A_{22} + A_{23} + A_{24} + A_{25} + A_{26} + A_{27} + A_{28} + A_{29} +$
$A_{30} + A_{31} + A_{32} + A_{33} + A_{34} + A_{35} + A_{36} + A_{37} + A_{38} +$
$A_{39} + A_{40})$

[add_assoc_50_var]

$\vdash \forall A_1\ A_2\ A_3\ A_4\ A_5\ A_6\ A_7\ A_8\ A_9\ A_{10}\ A_{11}\ A_{12}\ A_{13}\ A_{14}\ A_{15}\ A_{16}\ A_{17}$
$A_{18}\ A_{19}\ A_{20}\ A_{21}\ A_{22}\ A_{23}\ A_{24}\ A_{25}\ A_{26}\ A_{27}\ A_{28}\ A_{29}\ A_{30}\ A_{31}$
$A_{32}\ A_{33}\ A_{34}\ A_{35}\ A_{36}\ A_{37}\ A_{38}\ A_{39}\ A_{40}\ A_{41}\ A_{42}\ A_{43}\ A_{44}\ A_{45}$
$A_{46}\ A_{47}\ A_{48}\ A_{49}\ A_{50}.$
$A_1 \neq \texttt{PosInf} \land A_2 \neq \texttt{PosInf} \land A_3 \neq \texttt{PosInf} \land A_4 \neq \texttt{PosInf} \land$
$A_5 \neq \texttt{PosInf} \land A_6 \neq \texttt{PosInf} \land A_7 \neq \texttt{PosInf} \land A_8 \neq \texttt{PosInf} \land$
$A_9 \neq \texttt{PosInf} \land A_{10} \neq \texttt{PosInf} \land A_{11} \neq \texttt{PosInf} \land$
$A_{12} \neq \texttt{PosInf} \land A_{13} \neq \texttt{PosInf} \land A_{14} \neq \texttt{PosInf} \land$
$A_{15} \neq \texttt{PosInf} \land A_{16} \neq \texttt{PosInf} \land A_{17} \neq \texttt{PosInf} \land$
$A_{18} \neq \texttt{PosInf} \land A_{19} \neq \texttt{PosInf} \land A_{20} \neq \texttt{PosInf} \land$
$A_{21} \neq \texttt{PosInf} \land A_{22} \neq \texttt{PosInf} \land A_{23} \neq \texttt{PosInf} \land$
$A_{24} \neq \texttt{PosInf} \land A_{25} \neq \texttt{PosInf} \land A_{26} \neq \texttt{PosInf} \land$
$A_{27} \neq \texttt{PosInf} \land A_{28} \neq \texttt{PosInf} \land A_{29} \neq \texttt{PosInf} \land$
$A_{30} \neq \texttt{PosInf} \land A_{31} \neq \texttt{PosInf} \land A_{32} \neq \texttt{PosInf} \land$
$A_{33} \neq \texttt{PosInf} \land A_{34} \neq \texttt{PosInf} \land A_{35} \neq \texttt{PosInf} \land$
$A_{36} \neq \texttt{PosInf} \land A_{37} \neq \texttt{PosInf} \land A_{38} \neq \texttt{PosInf} \land$
$A_{39} \neq \texttt{PosInf} \land A_{40} \neq \texttt{PosInf} \land A_{41} \neq \texttt{PosInf} \land$
$A_{42} \neq \texttt{PosInf} \land A_{43} \neq \texttt{PosInf} \land A_{44} \neq \texttt{PosInf} \land$
$A_{45} \neq \texttt{PosInf} \land A_{46} \neq \texttt{PosInf} \land A_{47} \neq \texttt{PosInf} \land$
$A_{48} \neq \texttt{PosInf} \land A_{49} \neq \texttt{PosInf} \land A_{50} \neq \texttt{PosInf} \Rightarrow$
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($A_{43}$ +
($A_{44}$ +
($A_{45}$ +
($A_{46}$ +
($A_{47}$ +
($A_{48}$ +
($A_{49}$ +
$A_{50}$)))))))))))))))))))))))))))))))))))))))))
$A_1$ + $A_2$ + $A_3$ + $A_4$ + $A_5$ + $A_6$ + $A_7$ + $A_8$ + $A_9$ + $A_{10}$ + $A_{11}$ +
$A_{12}$ + $A_{13}$ + $A_{14}$ + $A_{15}$ + $A_{16}$ + $A_{17}$ + $A_{18}$ + $A_{19}$ + $A_{20}$ +

$A_{21}$ + $A_{22}$ + $A_{23}$ + $A_{24}$ + $A_{25}$ + $A_{26}$ + $A_{27}$ + $A_{28}$ + $A_{29}$ +
$A_{30}$ + $A_{31}$ + $A_{32}$ + $A_{33}$ + $A_{34}$ + $A_{35}$ + $A_{36}$ + $A_{37}$ + $A_{38}$ +
$A_{39}$ + $A_{40}$ + $A_{41}$ + $A_{42}$ + $A_{43}$ + $A_{44}$ + $A_{45}$ + $A_{46}$ + $A_{47}$ +
$A_{48}$ + $A_{49}$ + $A_{50}$)

[add_assoc_5_var]

$\vdash \forall A_1\ A_2\ A_3\ A_4\ A_5$.
$A_1 \neq$ PosInf $\wedge$ $A_2 \neq$ PosInf $\wedge$ $A_3 \neq$ PosInf $\wedge$ $A_4 \neq$ PosInf $\wedge$
$A_5 \neq$ PosInf $\Rightarrow$
($A_1$ + ($A_2$ + ($A_3$ + ($A_4$ + $A_5$))) = $A_1$ + $A_2$ + $A_3$ + $A_4$ + $A_5$)

[add_assoc_5_var1]

$\vdash \forall A_1\ A_2\ A_3\ A_4\ A_5$.
$A_1 \neq$ PosInf $\wedge$ $A_2 \neq$ PosInf $\wedge$ $A_3 \neq$ PosInf $\wedge$ $A_4 \neq$ PosInf $\wedge$
$A_5 \neq$ PosInf $\Rightarrow$
($A_1$ + ($A_2$ + $A_3$ + $A_4$ + $A_5$) = $A_1$ + $A_2$ + $A_3$ + $A_4$ + $A_5$)

[add_assoc_63_var]

$\vdash \forall A_1\ A_2\ A_3\ A_4\ A_5\ A_6\ A_7\ A_8\ A_9\ A_{10}\ A_{11}\ A_{12}\ A_{13}\ A_{14}\ A_{15}\ A_{16}\ A_{17}$
$A_{18}\ A_{19}\ A_{20}\ A_{21}\ A_{22}\ A_{23}\ A_{24}\ A_{25}\ A_{26}\ A_{27}\ A_{28}\ A_{29}\ A_{30}\ A_{31}$
$A_{32}\ A_{33}\ A_{34}\ A_{35}\ A_{36}\ A_{37}\ A_{38}\ A_{39}\ A_{40}\ A_{41}\ A_{42}\ A_{43}\ A_{44}\ A_{45}$
$A_{46}\ A_{47}\ A_{48}\ A_{49}\ A_{50}\ A_{51}\ A_{52}\ A_{53}\ A_{54}\ A_{55}\ A_{56}\ A_{57}\ A_{58}\ A_{59}$
$A_{60}\ A_{61}\ A_{62}\ A_{63}$.
$A_1 \neq$ PosInf $\wedge$ $A_2 \neq$ PosInf $\wedge$ $A_3 \neq$ PosInf $\wedge$ $A_4 \neq$ PosInf $\wedge$
$A_5 \neq$ PosInf $\wedge$ $A_6 \neq$ PosInf $\wedge$ $A_7 \neq$ PosInf $\wedge$ $A_8 \neq$ PosInf $\wedge$
$A_9 \neq$ PosInf $\wedge$ $A_{10} \neq$ PosInf $\wedge$ $A_{11} \neq$ PosInf $\wedge$
$A_{12} \neq$ PosInf $\wedge$ $A_{13} \neq$ PosInf $\wedge$ $A_{14} \neq$ PosInf $\wedge$
$A_{15} \neq$ PosInf $\wedge$ $A_{16} \neq$ PosInf $\wedge$ $A_{17} \neq$ PosInf $\wedge$
$A_{18} \neq$ PosInf $\wedge$ $A_{19} \neq$ PosInf $\wedge$ $A_{20} \neq$ PosInf $\wedge$
$A_{21} \neq$ PosInf $\wedge$ $A_{22} \neq$ PosInf $\wedge$ $A_{23} \neq$ PosInf $\wedge$
$A_{24} \neq$ PosInf $\wedge$ $A_{25} \neq$ PosInf $\wedge$ $A_{26} \neq$ PosInf $\wedge$
$A_{27} \neq$ PosInf $\wedge$ $A_{28} \neq$ PosInf $\wedge$ $A_{29} \neq$ PosInf $\wedge$
$A_{30} \neq$ PosInf $\wedge$ $A_{31} \neq$ PosInf $\wedge$ $A_{32} \neq$ PosInf $\wedge$
$A_{33} \neq$ PosInf $\wedge$ $A_{34} \neq$ PosInf $\wedge$ $A_{35} \neq$ PosInf $\wedge$
$A_{36} \neq$ PosInf $\wedge$ $A_{37} \neq$ PosInf $\wedge$ $A_{38} \neq$ PosInf $\wedge$
$A_{39} \neq$ PosInf $\wedge$ $A_{40} \neq$ PosInf $\wedge$ $A_{41} \neq$ PosInf $\wedge$
$A_{42} \neq$ PosInf $\wedge$ $A_{43} \neq$ PosInf $\wedge$ $A_{44} \neq$ PosInf $\wedge$
$A_{45} \neq$ PosInf $\wedge$ $A_{46} \neq$ PosInf $\wedge$ $A_{47} \neq$ PosInf $\wedge$
$A_{48} \neq$ PosInf $\wedge$ $A_{49} \neq$ PosInf $\wedge$ $A_{50} \neq$ PosInf $\wedge$
$A_{51} \neq$ PosInf $\wedge$ $A_{52} \neq$ PosInf $\wedge$ $A_{53} \neq$ PosInf $\wedge$
$A_{54} \neq$ PosInf $\wedge$ $A_{54} \neq$ PosInf $\wedge$ $A_{55} \neq$ PosInf $\wedge$
$A_{56} \neq$ PosInf $\wedge$ $A_{57} \neq$ PosInf $\wedge$ $A_{58} \neq$ PosInf $\wedge$
$A_{59} \neq$ PosInf $\wedge$ $A_{60} \neq$ PosInf $\wedge$ $A_{61} \neq$ PosInf $\wedge$
$A_{62} \neq$ PosInf $\wedge$ $A_{63} \neq$ PosInf $\Rightarrow$
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$(A_{61}$ +
$(A_{62}$ +
$A_{63}))))))))))))))))))))$

$A_1 + A_2 + A_3 + A_4 + A_5 + A_6 + A_7 + A_8 + A_9 + A_{10} + A_{11} +$
$A_{12} + A_{13} + A_{14} + A_{15} + A_{16} + A_{17} + A_{18} + A_{19} + A_{20} +$
$A_{21} + A_{22} + A_{23} + A_{24} + A_{25} + A_{26} + A_{27} + A_{28} + A_{29} +$
$A_{30} + A_{31} + A_{32} + A_{33} + A_{34} + A_{35} + A_{36} + A_{37} + A_{38} +$
$A_{39} + A_{40} + A_{41} + A_{42} + A_{43} + A_{44} + A_{45} + A_{46} + A_{47} +$
$A_{48} + A_{49} + A_{50} + A_{51} + A_{52} + A_{53} + A_{54} + A_{55} + A_{56} +$
$A_{57} + A_{58} + A_{59} + A_{60} + A_{61} + A_{62} + A_{63})$

[ALL_DISTINCT_RV_def]

⊢ ALL_DISTINCT_RV $[A_0;\ A_1;\ A_2;\ A_3;\ A_4;\ A_5;\ A_6;\ A_7;\ A_8;\ A_9]\ p\ t \iff$
ALL_DISTINCT
[DFT_event $p$ $A_0$ $t$; DFT_event $p$ $A_1$ $t$;
DFT_event $p$ (D_AND $A_2$ (D_BEFORE $A_3$ $A_2$)) $t$;
DFT_event $p$ (D_AND $A_2$ $A_4$) $t$; DFT_event $p$ (D_AND $A_5$ $A_6$) $t$;
DFT_event $p$ (D_AND (D_AND $A_7$ $A_8$) $A_9$) $t$] $\wedge$
rv_gt0_ninfinity $[A_0;\ A_1;\ A_2;\ A_3;\ A_4;\ A_5;\ A_6;\ A_7;\ A_8;\ A_9]\ \wedge$
$\forall s.$
ALL_DISTINCT
(MAP $(\lambda a.\ a\ s)$ $[A_0;\ A_1;\ A_2;\ A_3;\ A_4;\ A_5;\ A_6;\ A_7;\ A_8;\ A_9]$)

[ALL_DISTINCT_RV_ind]

⊢ $\forall P.$
$(\forall A_0\ A_1\ A_2\ A_3\ A_4\ A_5\ A_6\ A_7\ A_8\ A_9\ p\ t.$
$P\ [A_0;\ A_1;\ A_2;\ A_3;\ A_4;\ A_5;\ A_6;\ A_7;\ A_8;\ A_9]\ p\ t) \wedge$
$(\forall v_4.\ P$ [] (FST $v_4$) (SND $v_4$)) $\wedge$
$(\forall v_7\ v_8.\ P\ [v_7]$ (FST $v_8$) (SND $v_8$)) $\wedge$
$(\forall v_{12}\ v_{11}\ v_{13}.\ P\ [v_{12};\ v_{11}]$ (FST $v_{13}$) (SND $v_{13}$)) $\wedge$
$(\forall v_{18}\ v_{17}\ v_{16}\ v_{19}.\ P\ [v_{18};\ v_{17};\ v_{16}]$ (FST $v_{19}$) (SND $v_{19}$)) $\wedge$
$(\forall v_{25}\ v_{24}\ v_{23}\ v_{22}\ v_{26}.$
$P\ [v_{25};\ v_{24};\ v_{23};\ v_{22}]$ (FST $v_{26}$) (SND $v_{26}$)) $\wedge$
$(\forall v_{33}\ v_{32}\ v_{31}\ v_{30}\ v_{29}\ v_{34}.$
$P\ [v_{33};\ v_{32};\ v_{31};\ v_{30};\ v_{29}]$ (FST $v_{34}$) (SND $v_{34}$)) $\wedge$
$(\forall v_{42}\ v_{41}\ v_{40}\ v_{39}\ v_{38}\ v_{37}\ v_{43}.$
$P\ [v_{42};\ v_{41};\ v_{40};\ v_{39};\ v_{38};\ v_{37}]$ (FST $v_{43}$) (SND $v_{43}$)) $\wedge$
$(\forall v_{52}\ v_{51}\ v_{50}\ v_{49}\ v_{48}\ v_{47}\ v_{46}\ v_{53}.$
$P\ [v_{52};\ v_{51};\ v_{50};\ v_{49};\ v_{48};\ v_{47};\ v_{46}]$ (FST $v_{53}$)
(SND $v_{53}$)) $\wedge$
$(\forall v_{63}\ v_{62}\ v_{61}\ v_{60}\ v_{59}\ v_{58}\ v_{57}\ v_{56}\ v_{64}.$
$P\ [v_{63};\ v_{62};\ v_{61};\ v_{60};\ v_{59};\ v_{58};\ v_{57};\ v_{56}]$ (FST $v_{64}$)

$$\begin{array}{l}\quad\quad (\texttt{SND}\ v_{64})) \land \\ (\forall v_{75}\ v_{74}\ v_{73}\ v_{72}\ v_{71}\ v_{70}\ v_{69}\ v_{68}\ v_{67}\ v_{76}. \\ \quad P\ [v_{75};\ v_{74};\ v_{73};\ v_{72};\ v_{71};\ v_{70};\ v_{69};\ v_{68};\ v_{67}] \\ \quad\quad (\texttt{FST}\ v_{76})\ (\texttt{SND}\ v_{76})) \land \\ (\forall v_{92}\ v_{91}\ v_{90}\ v_{89}\ v_{88}\ v_{87}\ v_{86}\ v_{85}\ v_{84}\ v_{83}\ v_{79}\ v_{80}\ v_{93}. \\ \quad P \\ \quad\quad (v_{92}::v_{91}::v_{90}::v_{89}::v_{88}::v_{87}::v_{86}::v_{85}::v_{84}:: \\ \quad\quad\quad v_{83}::v_{79}::v_{80})\ (\texttt{FST}\ v_{93})\ (\texttt{SND}\ v_{93})) \Rightarrow \\ \forall v\ v_1\ v_2.\ P\ v\ v_1\ v_2\end{array}$$

[AND_OR_ID_LEFT]

$\vdash \forall A\ B.\ \texttt{D\_AND}\ A\ (\texttt{D\_OR}\ A\ B) = A$

[BIGINTER_10_sets]

$$\begin{array}{l}\vdash \forall x_1\ x_2\ x_3\ x_4\ x_5\ x_6\ x_7\ x_8\ x_9\ x_{10}. \\ \quad \texttt{BIGINTER}\ \{x_1;\ x_2;\ x_3;\ x_4;\ x_5;\ x_6;\ x_7;\ x_8;\ x_9;\ x_{10}\} = \\ \quad x_1 \cap x_2 \cap x_3 \cap x_4 \cap x_5 \cap x_6 \cap x_7 \cap x_8 \cap x_9 \cap x_{10}\end{array}$$

[BIGINTER_3_sets]

$\vdash \forall x\ y\ z.\ \texttt{BIGINTER}\ \{x;\ y;\ z\} = x \cap y \cap z$

[BIGINTER_4_sets]

$\vdash \forall x_1\ x_2\ x_3\ x_4.\ \texttt{BIGINTER}\ \{x_1;\ x_2;\ x_3;\ x_4\} = x_1 \cap x_2 \cap x_3 \cap x_4$

[BIGINTER_5_sets]

$$\begin{array}{l}\vdash \forall x_1\ x_2\ x_3\ x_4\ x_5. \\ \quad \texttt{BIGINTER}\ \{x_1;\ x_2;\ x_3;\ x_4;\ x_5\} = x_1 \cap x_2 \cap x_3 \cap x_4 \cap x_5\end{array}$$

[BIGINTER_6_sets]

$$\begin{array}{l}\vdash \forall x_1\ x_2\ x_3\ x_4\ x_5\ x_6. \\ \quad \texttt{BIGINTER}\ \{x_1;\ x_2;\ x_3;\ x_4;\ x_5;\ x_6\} = \\ \quad x_1 \cap x_2 \cap x_3 \cap x_4 \cap x_5 \cap x_6\end{array}$$

[BIGINTER_7_sets]

$$\begin{array}{l}\vdash \forall x_1\ x_2\ x_3\ x_4\ x_5\ x_6\ x_7. \\ \quad \texttt{BIGINTER}\ \{x_1;\ x_2;\ x_3;\ x_4;\ x_5;\ x_6;\ x_7\} = \\ \quad x_1 \cap x_2 \cap x_3 \cap x_4 \cap x_5 \cap x_6 \cap x_7\end{array}$$

[BIGINTER_8_sets]

$$\begin{array}{l}\vdash \forall x_1\ x_2\ x_3\ x_4\ x_5\ x_6\ x_7\ x_8. \\ \quad \texttt{BIGINTER}\ \{x_1;\ x_2;\ x_3;\ x_4;\ x_5;\ x_6;\ x_7;\ x_8\} = \\ \quad x_1 \cap x_2 \cap x_3 \cap x_4 \cap x_5 \cap x_6 \cap x_7 \cap x_8\end{array}$$

[BIGINTER_9_sets]

$$\begin{array}{l}\vdash \forall x_1\ x_2\ x_3\ x_4\ x_5\ x_6\ x_7\ x_8\ x_9. \\ \quad \texttt{BIGINTER}\ \{x_1;\ x_2;\ x_3;\ x_4;\ x_5;\ x_6;\ x_7;\ x_8;\ x_9\} = \\ \quad x_1 \cap x_2 \cap x_3 \cap x_4 \cap x_5 \cap x_6 \cap x_7 \cap x_8 \cap x_9\end{array}$$

[CAS_PROB]

```
⊢ ∀ CS SS MA MS MB P B PA PB PS p t f_MA.
       0 ≤ t ∧ prob_space p ∧
       ALL_DISTINCT_RV [CS; SS; MA; MS; MB; P; B; PA; PB; PS] p
         t ∧
       indep_vars_sets [CS; SS; MA; MS; MB; P; B; PA; PB; PS] p
         t ∧ distributed p lborel (λ s. real (MA s)) f_MA ∧
       (∀ y. 0 ≤ f_MA y) ∧ cont_CDF p (λ s. real (MS s)) ∧
       measurable_CDF p (λ s. real (MS s)) ⇒
       (prob p
          (DFT_event p
             (D_OR
                (D_OR
                   (D_AND (shared_spare PA PB PS PS)
                      (shared_spare PB PA PS PS))
                   (D_OR (P_AND MS MA) (HSP MA MB)))
                (HSP (FDEP (D_OR CS SS) P) (FDEP (D_OR CS SS) B)))
             t) =
        CDF p (λ s. real (CS s)) t + CDF p (λ s. real (SS s)) t +
        pos_fn_integral lborel
          (λ y.
                f_MA y ×
                (indicator_fn {u | 0 ≤ u ∧ u ≤ t} y ×
                 CDF p (λ s. real (MS s)) y)) +
        CDF p (λ s. real (MA s)) t × CDF p (λ s. real (MB s)) t +
        CDF p (λ s. real (P s)) t × CDF p (λ s. real (B s)) t +
        CDF p (λ s. real (PA s)) t × CDF p (λ s. real (PB s)) t ×
        CDF p (λ s. real (PS s)) t -
        CDF p (λ s. real (CS s)) t × CDF p (λ s. real (SS s)) t -
        CDF p (λ s. real (CS s)) t ×
        pos_fn_integral lborel
          (λ y.
                f_MA y ×
                (indicator_fn {u | 0 ≤ u ∧ u ≤ t} y ×
                 CDF p (λ s. real (MS s)) y)) -
        CDF p (λ s. real (CS s)) t × CDF p (λ s. real (MA s)) t ×
        CDF p (λ s. real (MB s)) t -
        CDF p (λ s. real (CS s)) t × CDF p (λ s. real (P s)) t ×
        CDF p (λ s. real (B s)) t -
        CDF p (λ s. real (CS s)) t × CDF p (λ s. real (PA s)) t ×
        CDF p (λ s. real (PB s)) t × CDF p (λ s. real (PS s)) t -
        CDF p (λ s. real (SS s)) t ×
        pos_fn_integral lborel
          (λ y.
                f_MA y ×
                (indicator_fn {u | 0 ≤ u ∧ u ≤ t} y ×
                 CDF p (λ s. real (MS s)) y)) -
        CDF p (λ s. real (SS s)) t × CDF p (λ s. real (MA s)) t ×
        CDF p (λ s. real (MB s)) t -
```

```
CDF p (λs. real (SS s)) t × CDF p (λs. real (P s)) t ×
CDF p (λs. real (B s)) t -
CDF p (λs. real (SS s)) t × CDF p (λs. real (PA s)) t ×
CDF p (λs. real (PB s)) t × CDF p (λs. real (PS s)) t -
pos_fn_integral lborel
  (λy.
       f_MA y ×
       (indicator_fn {u | 0 ≤ u ∧ u ≤ t} y ×
        CDF p (λs. real (MS s)) y)) ×
CDF p (λs. real (MB s)) t -
pos_fn_integral lborel
  (λy.
       f_MA y ×
       (indicator_fn {u | 0 ≤ u ∧ u ≤ t} y ×
        CDF p (λs. real (MS s)) y)) ×
CDF p (λs. real (P s)) t × CDF p (λs. real (B s)) t -
pos_fn_integral lborel
  (λy.
       f_MA y ×
       (indicator_fn {u | 0 ≤ u ∧ u ≤ t} y ×
        CDF p (λs. real (MS s)) y)) ×
CDF p (λs. real (PA s)) t × CDF p (λs. real (PB s)) t ×
CDF p (λs. real (PS s)) t -
CDF p (λs. real (MA s)) t × CDF p (λs. real (MB s)) t ×
CDF p (λs. real (P s)) t × CDF p (λs. real (B s)) t -
CDF p (λs. real (MA s)) t × CDF p (λs. real (MB s)) t ×
CDF p (λs. real (PA s)) t × CDF p (λs. real (PB s)) t ×
CDF p (λs. real (PS s)) t -
CDF p (λs. real (P s)) t × CDF p (λs. real (B s)) t ×
CDF p (λs. real (PA s)) t × CDF p (λs. real (PB s)) t ×
CDF p (λs. real (PS s)) t +
CDF p (λs. real (CS s)) t × CDF p (λs. real (SS s)) t ×
pos_fn_integral lborel
  (λy.
       f_MA y ×
       (indicator_fn {u | 0 ≤ u ∧ u ≤ t} y ×
        CDF p (λs. real (MS s)) y)) +
CDF p (λs. real (CS s)) t × CDF p (λs. real (SS s)) t ×
CDF p (λs. real (MA s)) t × CDF p (λs. real (MB s)) t +
CDF p (λs. real (CS s)) t × CDF p (λs. real (SS s)) t ×
CDF p (λs. real (P s)) t × CDF p (λs. real (B s)) t +
CDF p (λs. real (CS s)) t × CDF p (λs. real (SS s)) t ×
CDF p (λs. real (PA s)) t × CDF p (λs. real (PB s)) t ×
CDF p (λs. real (PS s)) t +
CDF p (λs. real (CS s)) t ×
pos_fn_integral lborel
  (λy.
       f_MA y ×
       (indicator_fn {u | 0 ≤ u ∧ u ≤ t} y ×
```

```
       CDF p (λs. real (MS s)) y)) ×
CDF p (λs. real (MB s)) t +
CDF p (λs. real (CS s)) t ×
pos_fn_integral lborel
  (λy.
      f_MA y ×
      (indicator_fn {u | 0 ≤ u ∧ u ≤ t} y ×
       CDF p (λs. real (MS s)) y)) ×
CDF p (λs. real (P s)) t × CDF p (λs. real (B s)) t +
CDF p (λs. real (CS s)) t ×
pos_fn_integral lborel
  (λy.
      f_MA y ×
      (indicator_fn {u | 0 ≤ u ∧ u ≤ t} y ×
       CDF p (λs. real (MS s)) y)) ×
CDF p (λs. real (PA s)) t × CDF p (λs. real (PB s)) t ×
CDF p (λs. real (PS s)) t +
CDF p (λs. real (CS s)) t × CDF p (λs. real (MA s)) t ×
CDF p (λs. real (MB s)) t × CDF p (λs. real (P s)) t ×
CDF p (λs. real (B s)) t +
CDF p (λs. real (CS s)) t × CDF p (λs. real (MA s)) t ×
CDF p (λs. real (MB s)) t × CDF p (λs. real (PA s)) t ×
CDF p (λs. real (PB s)) t × CDF p (λs. real (PS s)) t +
CDF p (λs. real (CS s)) t × CDF p (λs. real (P s)) t ×
CDF p (λs. real (B s)) t × CDF p (λs. real (PA s)) t ×
CDF p (λs. real (PB s)) t × CDF p (λs. real (PS s)) t +
CDF p (λs. real (SS s)) t ×
pos_fn_integral lborel
  (λy.
      f_MA y ×
      (indicator_fn {u | 0 ≤ u ∧ u ≤ t} y ×
       CDF p (λs. real (MS s)) y)) ×
CDF p (λs. real (MB s)) t +
CDF p (λs. real (SS s)) t ×
pos_fn_integral lborel
  (λy.
      f_MA y ×
      (indicator_fn {u | 0 ≤ u ∧ u ≤ t} y ×
       CDF p (λs. real (MS s)) y)) ×
CDF p (λs. real (P s)) t × CDF p (λs. real (B s)) t +
CDF p (λs. real (SS s)) t ×
pos_fn_integral lborel
  (λy.
      f_MA y ×
      (indicator_fn {u | 0 ≤ u ∧ u ≤ t} y ×
       CDF p (λs. real (MS s)) y)) ×
CDF p (λs. real (PA s)) t × CDF p (λs. real (PB s)) t ×
CDF p (λs. real (PS s)) t +
CDF p (λs. real (SS s)) t × CDF p (λs. real (MA s)) t ×
```

```
CDF p (λs. real (MB s)) t × CDF p (λs. real (P s)) t ×
CDF p (λs. real (B s)) t +
CDF p (λs. real (SS s)) t × CDF p (λs. real (MA s)) t ×
CDF p (λs. real (MB s)) t × CDF p (λs. real (PA s)) t ×
CDF p (λs. real (PB s)) t × CDF p (λs. real (PS s)) t +
CDF p (λs. real (SS s)) t × CDF p (λs. real (P s)) t ×
CDF p (λs. real (B s)) t × CDF p (λs. real (PA s)) t ×
CDF p (λs. real (PB s)) t × CDF p (λs. real (PS s)) t +
pos_fn_integral lborel
  (λy.
       f_MA y ×
       (indicator_fn {u | 0 ≤ u ∧ u ≤ t} y ×
        CDF p (λs. real (MS s)) y)) ×
CDF p (λs. real (MB s)) t × CDF p (λs. real (P s)) t ×
CDF p (λs. real (B s)) t +
pos_fn_integral lborel
  (λy.
       f_MA y ×
       (indicator_fn {u | 0 ≤ u ∧ u ≤ t} y ×
        CDF p (λs. real (MS s)) y)) ×
CDF p (λs. real (MB s)) t × CDF p (λs. real (PA s)) t ×
CDF p (λs. real (PB s)) t × CDF p (λs. real (PS s)) t +
pos_fn_integral lborel
  (λy.
       f_MA y ×
       (indicator_fn {u | 0 ≤ u ∧ u ≤ t} y ×
        CDF p (λs. real (MS s)) y)) ×
CDF p (λs. real (P s)) t × CDF p (λs. real (B s)) t ×
CDF p (λs. real (PA s)) t × CDF p (λs. real (PB s)) t ×
CDF p (λs. real (PS s)) t +
CDF p (λs. real (MA s)) t × CDF p (λs. real (MB s)) t ×
CDF p (λs. real (P s)) t × CDF p (λs. real (B s)) t ×
CDF p (λs. real (PA s)) t × CDF p (λs. real (PB s)) t ×
CDF p (λs. real (PS s)) t -
CDF p (λs. real (CS s)) t × CDF p (λs. real (SS s)) t ×
pos_fn_integral lborel
  (λy.
       f_MA y ×
       (indicator_fn {u | 0 ≤ u ∧ u ≤ t} y ×
        CDF p (λs. real (MS s)) y)) ×
CDF p (λs. real (MB s)) t -
CDF p (λs. real (CS s)) t × CDF p (λs. real (SS s)) t ×
pos_fn_integral lborel
  (λy.
       f_MA y ×
       (indicator_fn {u | 0 ≤ u ∧ u ≤ t} y ×
        CDF p (λs. real (MS s)) y)) ×
CDF p (λs. real (P s)) t × CDF p (λs. real (B s)) t -
CDF p (λs. real (CS s)) t × CDF p (λs. real (SS s)) t ×
```

```
pos_fn_integral lborel
  (λ y.
      f_MA y ×
      (indicator_fn {u | 0 ≤ u ∧ u ≤ t} y ×
       CDF p (λ s. real (MS s)) y)) ×
CDF p (λ s. real (PA s)) t × CDF p (λ s. real (PB s)) t ×
CDF p (λ s. real (PS s)) t -
CDF p (λ s. real (CS s)) t × CDF p (λ s. real (SS s)) t ×
CDF p (λ s. real (MA s)) t × CDF p (λ s. real (MB s)) t ×
CDF p (λ s. real (P s)) t × CDF p (λ s. real (B s)) t -
CDF p (λ s. real (CS s)) t × CDF p (λ s. real (SS s)) t ×
CDF p (λ s. real (MA s)) t × CDF p (λ s. real (MB s)) t ×
CDF p (λ s. real (PA s)) t × CDF p (λ s. real (PB s)) t ×
CDF p (λ s. real (PS s)) t -
CDF p (λ s. real (CS s)) t × CDF p (λ s. real (SS s)) t ×
CDF p (λ s. real (P s)) t × CDF p (λ s. real (B s)) t ×
CDF p (λ s. real (PA s)) t × CDF p (λ s. real (PB s)) t ×
CDF p (λ s. real (PS s)) t -
CDF p (λ s. real (CS s)) t ×
pos_fn_integral lborel
  (λ y.
      f_MA y ×
      (indicator_fn {u | 0 ≤ u ∧ u ≤ t} y ×
       CDF p (λ s. real (MS s)) y)) ×
CDF p (λ s. real (MB s)) t × CDF p (λ s. real (P s)) t ×
CDF p (λ s. real (B s)) t -
CDF p (λ s. real (CS s)) t ×
pos_fn_integral lborel
  (λ y.
      f_MA y ×
      (indicator_fn {u | 0 ≤ u ∧ u ≤ t} y ×
       CDF p (λ s. real (MS s)) y)) ×
CDF p (λ s. real (MB s)) t × CDF p (λ s. real (PA s)) t ×
CDF p (λ s. real (PB s)) t × CDF p (λ s. real (PS s)) t -
CDF p (λ s. real (CS s)) t ×
pos_fn_integral lborel
  (λ y.
      f_MA y ×
      (indicator_fn {u | 0 ≤ u ∧ u ≤ t} y ×
       CDF p (λ s. real (MS s)) y)) ×
CDF p (λ s. real (P s)) t × CDF p (λ s. real (B s)) t ×
CDF p (λ s. real (PA s)) t × CDF p (λ s. real (PB s)) t ×
CDF p (λ s. real (PS s)) t -
CDF p (λ s. real (CS s)) t × CDF p (λ s. real (MA s)) t ×
CDF p (λ s. real (MB s)) t × CDF p (λ s. real (P s)) t ×
CDF p (λ s. real (B s)) t × CDF p (λ s. real (PA s)) t ×
CDF p (λ s. real (PB s)) t × CDF p (λ s. real (PS s)) t -
CDF p (λ s. real (SS s)) t ×
pos_fn_integral lborel
```

```
  (λ y.
       f_MA y ×
       (indicator_fn {u | 0 ≤ u ∧ u ≤ t} y ×
        CDF p (λ s. real (MS s)) y)) ×
CDF p (λ s. real (MB s)) t × CDF p (λ s. real (P s)) t ×
CDF p (λ s. real (B s)) t -
CDF p (λ s. real (SS s)) t ×
pos_fn_integral lborel
  (λ y.
       f_MA y ×
       (indicator_fn {u | 0 ≤ u ∧ u ≤ t} y ×
        CDF p (λ s. real (MS s)) y)) ×
CDF p (λ s. real (MB s)) t × CDF p (λ s. real (PA s)) t ×
CDF p (λ s. real (PB s)) t × CDF p (λ s. real (PS s)) t -
CDF p (λ s. real (SS s)) t ×
pos_fn_integral lborel
  (λ y.
       f_MA y ×
       (indicator_fn {u | 0 ≤ u ∧ u ≤ t} y ×
        CDF p (λ s. real (MS s)) y)) ×
CDF p (λ s. real (P s)) t × CDF p (λ s. real (B s)) t ×
CDF p (λ s. real (PA s)) t × CDF p (λ s. real (PB s)) t ×
CDF p (λ s. real (PS s)) t -
CDF p (λ s. real (SS s)) t × CDF p (λ s. real (MA s)) t ×
CDF p (λ s. real (MB s)) t × CDF p (λ s. real (P s)) t ×
CDF p (λ s. real (B s)) t × CDF p (λ s. real (PA s)) t ×
CDF p (λ s. real (PB s)) t × CDF p (λ s. real (PS s)) t -
pos_fn_integral lborel
  (λ y.
       f_MA y ×
       (indicator_fn {u | 0 ≤ u ∧ u ≤ t} y ×
        CDF p (λ s. real (MS s)) y)) ×
CDF p (λ s. real (MB s)) t × CDF p (λ s. real (P s)) t ×
CDF p (λ s. real (B s)) t × CDF p (λ s. real (PA s)) t ×
CDF p (λ s. real (PB s)) t × CDF p (λ s. real (PS s)) t +
CDF p (λ s. real (CS s)) t × CDF p (λ s. real (SS s)) t ×
pos_fn_integral lborel
  (λ y.
       f_MA y ×
       (indicator_fn {u | 0 ≤ u ∧ u ≤ t} y ×
        CDF p (λ s. real (MS s)) y)) ×
CDF p (λ s. real (MB s)) t × CDF p (λ s. real (P s)) t ×
CDF p (λ s. real (B s)) t +
CDF p (λ s. real (CS s)) t × CDF p (λ s. real (SS s)) t ×
pos_fn_integral lborel
  (λ y.
       f_MA y ×
       (indicator_fn {u | 0 ≤ u ∧ u ≤ t} y ×
        CDF p (λ s. real (MS s)) y)) ×
```

```
CDF p (λs. real (MB s)) t × CDF p (λs. real (PA s)) t ×
CDF p (λs. real (PB s)) t × CDF p (λs. real (PS s)) t +
CDF p (λs. real (CS s)) t × CDF p (λs. real (SS s)) t ×
pos_fn_integral lborel
  (λy.
      f_MA y ×
      (indicator_fn {u | 0 ≤ u ∧ u ≤ t} y ×
       CDF p (λs. real (MS s)) y)) ×
CDF p (λs. real (P s)) t × CDF p (λs. real (B s)) t ×
CDF p (λs. real (PA s)) t × CDF p (λs. real (PB s)) t ×
CDF p (λs. real (PS s)) t +
CDF p (λs. real (CS s)) t × CDF p (λs. real (SS s)) t ×
CDF p (λs. real (MA s)) t × CDF p (λs. real (MB s)) t ×
CDF p (λs. real (P s)) t × CDF p (λs. real (B s)) t ×
CDF p (λs. real (PA s)) t × CDF p (λs. real (PB s)) t ×
CDF p (λs. real (PS s)) t +
CDF p (λs. real (CS s)) t ×
pos_fn_integral lborel
  (λy.
      f_MA y ×
      (indicator_fn {u | 0 ≤ u ∧ u ≤ t} y ×
       CDF p (λs. real (MS s)) y)) ×
CDF p (λs. real (MB s)) t × CDF p (λs. real (P s)) t ×
CDF p (λs. real (B s)) t × CDF p (λs. real (PA s)) t ×
CDF p (λs. real (PB s)) t × CDF p (λs. real (PS s)) t +
CDF p (λs. real (SS s)) t ×
pos_fn_integral lborel
  (λy.
      f_MA y ×
      (indicator_fn {u | 0 ≤ u ∧ u ≤ t} y ×
       CDF p (λs. real (MS s)) y)) ×
CDF p (λs. real (MB s)) t × CDF p (λs. real (P s)) t ×
CDF p (λs. real (B s)) t × CDF p (λs. real (PA s)) t ×
CDF p (λs. real (PB s)) t × CDF p (λs. real (PS s)) t -
CDF p (λs. real (CS s)) t × CDF p (λs. real (SS s)) t ×
pos_fn_integral lborel
  (λy.
      f_MA y ×
      (indicator_fn {u | 0 ≤ u ∧ u ≤ t} y ×
       CDF p (λs. real (MS s)) y)) ×
CDF p (λs. real (MB s)) t × CDF p (λs. real (P s)) t ×
CDF p (λs. real (B s)) t × CDF p (λs. real (PA s)) t ×
CDF p (λs. real (PB s)) t × CDF p (λs. real (PS s)) t)
```

[CAS_UNION_LIST]

```
⊢ ∀PA PB PS MS MA MB CS SS P B p t.
      DFT_event p
        (D_OR
           (D_OR
```

```
            (D_OR
               (D_OR (D_OR CS SS) (D_AND MA (D_BEFORE MS MA)))
               (D_AND MA MB)) (D_AND P B))
         (D_AND (D_AND PA PB) PS)) t =
      union_list
        [DFT_event p CS t; DFT_event p SS t;
         DFT_event p (D_AND MA (D_BEFORE MS MA)) t;
         DFT_event p (D_AND MA MB) t;
         DFT_event p (D_AND P B) t;
         DFT_event p (D_AND (D_AND PA PB) PS) t]
```

[disjoint_and_before_inter_event]

```
 ⊢ ∀A B C p t.
      DISJOINT
        ({s | A s < B s ∧ B s ≤ Normal t} ∩ DFT_event p C t)
        ({s | B s ≤ A s ∧ PosInf ≤ Normal t} ∩ DFT_event p C t)
```

[event_and_before_union]

```
 ⊢ ∀A B C p t.
      DFT_event p (D_AND B (D_BEFORE A B)) t ∩ DFT_event p C t =
      {s | A s < B s ∧ B s ≤ Normal t} ∩ DFT_event p C t ∪
      {s | B s ≤ A s ∧ PosInf ≤ Normal t} ∩ DFT_event p C t
```

[event_before_union_event]

```
 ⊢ ∀A B C p t.
      DFT_event p (D_BEFORE A B) t ∩ DFT_event p C t =
      {s | A s < B s ∧ A s ≤ Normal t} ∩ DFT_event p C t ∪
      {s | B s ≤ A s ∧ PosInf ≤ Normal t} ∩ DFT_event p C t
```

[EVENTS_INTER_3]

```
 ⊢ ∀A1 A2 A3 p.
      prob_space p ∧ A1 ∈ events p ∧ A2 ∈ events p ∧
      A3 ∈ events p ⇒
      A1 ∩ A2 ∩ A3 ∈ events p
```

[EVENTS_INTER_4]

```
 ⊢ ∀A1 A2 A3 A4 p.
      prob_space p ∧ A1 ∈ events p ∧ A2 ∈ events p ∧
      A3 ∈ events p ∧ A4 ∈ events p ⇒
      A1 ∩ A2 ∩ A3 ∩ A4 ∈ events p
```

[EVENTS_INTER_5]

```
 ⊢ ∀A1 A2 A3 A4 A5 p.
      prob_space p ∧ A1 ∈ events p ∧ A2 ∈ events p ∧
      A3 ∈ events p ∧ A4 ∈ events p ∧ A5 ∈ events p ⇒
      A1 ∩ A2 ∩ A3 ∩ A4 ∩ A5 ∈ events p
```

[EVENTS_INTER_6]

$$\vdash \forall A_1\ A_2\ A_3\ A_4\ A_5\ A_6\ p.\ \texttt{prob\_space}\ p \wedge A_1 \in \texttt{events}\ p \wedge A_2 \in \texttt{events}\ p \wedge A_3 \in \texttt{events}\ p \wedge A_4 \in \texttt{events}\ p \wedge A_5 \in \texttt{events}\ p \wedge A_6 \in \texttt{events}\ p \Rightarrow A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5 \cap A_6 \in \texttt{events}\ p$$

[extreal_add_sub_2_var]

$$\vdash \forall x\ y.\ x \neq \texttt{PosInf} \wedge y \neq \texttt{PosInf} \wedge x \neq \texttt{NegInf} \wedge y \neq \texttt{NegInf} \Rightarrow x - y \neq \texttt{PosInf} \wedge x - y \neq \texttt{NegInf} \wedge -x - y \neq \texttt{PosInf} \wedge -x - y \neq \texttt{NegInf} \wedge -x + -y \neq \texttt{PosInf} \wedge -x + -y \neq \texttt{NegInf}$$

[extreal_add_sub_63]

$$\vdash \forall A_1\ A_2\ A_3\ A_4\ A_5\ A_6\ A_7\ A_8\ A_9\ A_{10}\ A_{11}\ A_{12}\ A_{13}\ A_{14}\ A_{15}\ A_{16}\ A_{17}\ A_{18}\ A_{19}\ A_{20}\ A_{21}\ A_{22}\ A_{23}\ A_{24}\ A_{25}\ A_{26}\ A_{27}\ A_{28}\ A_{29}\ A_{30}\ A_{31}\ A_{32}\ A_{33}\ A_{34}\ A_{35}\ A_{36}\ A_{37}\ A_{38}\ A_{39}\ A_{40}\ A_{41}\ A_{42}\ A_{43}\ A_{44}\ A_{45}\ A_{46}\ A_{47}\ A_{48}\ A_{49}\ A_{50}\ A_{51}\ A_{52}\ A_{53}\ A_{54}\ A_{55}\ A_{56}\ A_{57}\ A_{58}\ A_{59}\ A_{60}\ A_{61}\ A_{62}\ A_{63}.$$

$$A_1 \neq \texttt{PosInf} \wedge A_1 \neq \texttt{NegInf} \wedge A_2 \neq \texttt{PosInf} \wedge A_2 \neq \texttt{NegInf} \wedge$$
$$A_3 \neq \texttt{PosInf} \wedge A_3 \neq \texttt{NegInf} \wedge A_4 \neq \texttt{PosInf} \wedge A_4 \neq \texttt{NegInf} \wedge$$
$$A_5 \neq \texttt{PosInf} \wedge A_5 \neq \texttt{NegInf} \wedge A_6 \neq \texttt{PosInf} \wedge A_6 \neq \texttt{NegInf} \wedge$$
$$A_7 \neq \texttt{PosInf} \wedge A_7 \neq \texttt{NegInf} \wedge A_8 \neq \texttt{PosInf} \wedge A_8 \neq \texttt{NegInf} \wedge$$
$$A_9 \neq \texttt{PosInf} \wedge A_9 \neq \texttt{NegInf} \wedge A_{10} \neq \texttt{PosInf} \wedge A_{10} \neq \texttt{NegInf} \wedge$$
$$A_{11} \neq \texttt{PosInf} \wedge A_{11} \neq \texttt{NegInf} \wedge A_{12} \neq \texttt{PosInf} \wedge$$
$$A_{12} \neq \texttt{NegInf} \wedge A_{13} \neq \texttt{PosInf} \wedge A_{13} \neq \texttt{NegInf} \wedge$$
$$A_{14} \neq \texttt{PosInf} \wedge A_{14} \neq \texttt{NegInf} \wedge A_{15} \neq \texttt{PosInf} \wedge$$
$$A_{15} \neq \texttt{NegInf} \wedge A_{16} \neq \texttt{PosInf} \wedge A_{16} \neq \texttt{NegInf} \wedge$$
$$A_{17} \neq \texttt{PosInf} \wedge A_{17} \neq \texttt{NegInf} \wedge A_{18} \neq \texttt{PosInf} \wedge$$
$$A_{18} \neq \texttt{NegInf} \wedge A_{19} \neq \texttt{PosInf} \wedge A_{19} \neq \texttt{NegInf} \wedge$$
$$A_{20} \neq \texttt{PosInf} \wedge A_{20} \neq \texttt{NegInf} \wedge A_{21} \neq \texttt{PosInf} \wedge$$
$$A_{21} \neq \texttt{NegInf} \wedge A_{22} \neq \texttt{PosInf} \wedge A_{22} \neq \texttt{NegInf} \wedge$$
$$A_{23} \neq \texttt{PosInf} \wedge A_{23} \neq \texttt{NegInf} \wedge A_{24} \neq \texttt{PosInf} \wedge$$
$$A_{24} \neq \texttt{NegInf} \wedge A_{25} \neq \texttt{PosInf} \wedge A_{25} \neq \texttt{NegInf} \wedge$$
$$A_{26} \neq \texttt{PosInf} \wedge A_{25} \neq \texttt{NegInf} \wedge A_{26} \neq \texttt{PosInf} \wedge$$
$$A_{26} \neq \texttt{NegInf} \wedge A_{27} \neq \texttt{PosInf} \wedge A_{27} \neq \texttt{NegInf} \wedge$$
$$A_{28} \neq \texttt{PosInf} \wedge A_{28} \neq \texttt{NegInf} \wedge A_{29} \neq \texttt{PosInf} \wedge$$
$$A_{29} \neq \texttt{NegInf} \wedge A_{30} \neq \texttt{PosInf} \wedge A_{30} \neq \texttt{NegInf} \wedge$$
$$A_{31} \neq \texttt{PosInf} \wedge A_{31} \neq \texttt{NegInf} \wedge A_{32} \neq \texttt{PosInf} \wedge$$
$$A_{32} \neq \texttt{NegInf} \wedge A_{33} \neq \texttt{PosInf} \wedge A_{33} \neq \texttt{NegInf} \wedge$$
$$A_{34} \neq \texttt{PosInf} \wedge A_{34} \neq \texttt{NegInf} \wedge A_{35} \neq \texttt{PosInf} \wedge$$
$$A_{35} \neq \texttt{NegInf} \wedge A_{36} \neq \texttt{PosInf} \wedge A_{36} \neq \texttt{NegInf} \wedge$$
$$A_{37} \neq \texttt{PosInf} \wedge A_{37} \neq \texttt{NegInf} \wedge A_{38} \neq \texttt{PosInf} \wedge$$
$$A_{38} \neq \texttt{NegInf} \wedge A_{39} \neq \texttt{PosInf} \wedge A_{39} \neq \texttt{NegInf} \wedge$$
$$A_{40} \neq \texttt{PosInf} \wedge A_{40} \neq \texttt{NegInf} \wedge A_{41} \neq \texttt{PosInf} \wedge$$
$$A_{41} \neq \texttt{NegInf} \wedge A_{42} \neq \texttt{PosInf} \wedge A_{42} \neq \texttt{NegInf} \wedge$$
$$A_{43} \neq \texttt{PosInf} \wedge A_{43} \neq \texttt{NegInf} \wedge A_{44} \neq \texttt{PosInf} \wedge$$
$$A_{44} \neq \texttt{NegInf} \wedge A_{45} \neq \texttt{PosInf} \wedge A_{45} \neq \texttt{NegInf} \wedge$$

$A_{46}$ ≠ PosInf ∧ $A_{46}$ ≠ NegInf ∧ $A_{47}$ ≠ PosInf ∧
$A_{47}$ ≠ NegInf ∧ $A_{48}$ ≠ PosInf ∧ $A_{48}$ ≠ NegInf ∧
$A_{49}$ ≠ PosInf ∧ $A_{49}$ ≠ NegInf ∧ $A_{50}$ ≠ PosInf ∧
$A_{50}$ ≠ NegInf ∧ $A_{51}$ ≠ PosInf ∧ $A_{51}$ ≠ NegInf ∧
$A_{52}$ ≠ PosInf ∧ $A_{52}$ ≠ NegInf ∧ $A_{53}$ ≠ PosInf ∧
$A_{53}$ ≠ NegInf ∧ $A_{54}$ ≠ PosInf ∧ $A_{54}$ ≠ NegInf ∧
$A_{55}$ ≠ PosInf ∧ $A_{55}$ ≠ NegInf ∧ $A_{56}$ ≠ PosInf ∧
$A_{56}$ ≠ NegInf ∧ $A_{57}$ ≠ PosInf ∧ $A_{57}$ ≠ NegInf ∧
$A_{58}$ ≠ PosInf ∧ $A_{58}$ ≠ NegInf ∧ $A_{59}$ ≠ PosInf ∧
$A_{59}$ ≠ NegInf ∧ $A_{60}$ ≠ PosInf ∧ $A_{60}$ ≠ NegInf ∧
$A_{61}$ ≠ PosInf ∧ $A_{61}$ ≠ NegInf ∧ $A_{62}$ ≠ PosInf ∧
$A_{62}$ ≠ NegInf ∧ $A_{63}$ ≠ PosInf ∧ $A_{63}$ ≠ NegInf ⇒
($A_1$ + $A_2$ + $A_3$ + $A_4$ + $A_5$ + $A_6$ + -$A_7$ + -$A_8$ + -$A_9$ + -$A_{10}$ +
-$A_{11}$ + -$A_{12}$ + -$A_{13}$ + -$A_{14}$ + -$A_{15}$ + -$A_{16}$ + -$A_{17}$ + -$A_{18}$ +
-$A_{19}$ + -$A_{20}$ + -$A_{21}$ + $A_{22}$ + $A_{23}$ + $A_{24}$ + $A_{25}$ + $A_{26}$ + $A_{27}$ +
$A_{28}$ + $A_{29}$ + $A_{30}$ + $A_{31}$ + $A_{32}$ + $A_{33}$ + $A_{34}$ + $A_{35}$ + $A_{36}$ +
$A_{37}$ + $A_{38}$ + $A_{39}$ + $A_{40}$ + $A_{41}$ + -$A_{42}$ + -$A_{43}$ + -$A_{44}$ +
-$A_{45}$ + -$A_{46}$ + -$A_{47}$ + -$A_{48}$ + -$A_{49}$ + -$A_{50}$ + -$A_{51}$ + -$A_{52}$ +
-$A_{53}$ + -$A_{54}$ + -$A_{55}$ + -$A_{56}$ + $A_{57}$ + $A_{58}$ + $A_{59}$ + $A_{60}$ +
$A_{61}$ + $A_{62}$ + -$A_{63}$ =
$A_1$ + $A_2$ + $A_3$ + $A_4$ + $A_5$ + $A_6$ - $A_7$ - $A_8$ - $A_9$ - $A_{10}$ - $A_{11}$ -
$A_{12}$ - $A_{13}$ - $A_{14}$ - $A_{15}$ - $A_{16}$ - $A_{17}$ - $A_{18}$ - $A_{19}$ - $A_{20}$ -
$A_{21}$ + $A_{22}$ + $A_{23}$ + $A_{24}$ + $A_{25}$ + $A_{26}$ + $A_{27}$ + $A_{28}$ + $A_{29}$ +
$A_{30}$ + $A_{31}$ + $A_{32}$ + $A_{33}$ + $A_{34}$ + $A_{35}$ + $A_{36}$ + $A_{37}$ + $A_{38}$ +
$A_{39}$ + $A_{40}$ + $A_{41}$ - $A_{42}$ - $A_{43}$ - $A_{44}$ - $A_{45}$ - $A_{46}$ - $A_{47}$ -
$A_{48}$ - $A_{49}$ - $A_{50}$ - $A_{51}$ - $A_{52}$ - $A_{53}$ - $A_{54}$ - $A_{55}$ - $A_{56}$ +
$A_{57}$ + $A_{58}$ + $A_{59}$ + $A_{60}$ + $A_{61}$ + $A_{62}$ - $A_{63}$)

[extreal_sub_sub]

⊢ ∀$x$ $y$.
$x$ ≠ PosInf ∧ $x$ ≠ NegInf ∧ $y$ ≠ PosInf ∧ $y$ ≠ NegInf ⇒
(-$x$ + -$y$ = -$x$ - $y$)

[HCAS]

```
⊢ ∀PA PB PS MS MA MB CS SS P B.
    (∀s. ALL_DISTINCT [MA s; MS s; PA s; PB s; PS s]) ⇒
    (D_OR
       (D_OR
          (D_AND (shared_spare PA PB PS PS)
             (shared_spare PB PA PS PS))
          (D_OR (P_AND MS MA) (HSP MA MB)))
       (HSP (FDEP (D_OR CS SS) P) (FDEP (D_OR CS SS) B)) =
     D_OR
       (D_OR
          (D_OR (D_OR CS SS)
             (D_OR (P_AND MS MA) (D_AND MA MB))) (D_AND P B))
       (D_AND (D_AND PA PB) PS))
```

[HCAS_final]

```
⊢ ∀PA PB PS MS MA MB CS SS P B.
      (∀s. ALL_DISTINCT [MA s; MS s; PA s; PB s; PS s]) ⇒
      (D_OR
         (D_OR
            (D_AND (shared_spare PA PB PS PS)
               (shared_spare PB PA PS PS))
            (D_OR (P_AND MS MA) (HSP MA MB)))
         (HSP (FDEP (D_OR CS SS) P) (FDEP (D_OR CS SS) B)) =
       D_OR
         (D_OR
            (D_OR (D_OR CS SS)
               (D_OR (D_AND MA (D_BEFORE MS MA)) (D_AND MA MB)))
            (D_AND P B)) (D_AND (D_AND PA PB) PS))
```

[HCAS_final_2]

```
⊢ ∀PA PB PS MS MA MB CS SS P B.
      (∀s. ALL_DISTINCT [MA s; MS s; PA s; PB s; PS s]) ⇒
      (D_OR
         (D_OR
            (D_AND (shared_spare PA PB PS PS)
               (shared_spare PB PA PS PS))
            (D_OR (P_AND MS MA) (HSP MA MB)))
         (HSP (FDEP (D_OR CS SS) P) (FDEP (D_OR CS SS) B)) =
       D_OR
         (D_OR
            (D_OR
               (D_OR (D_OR CS SS) (D_AND MA (D_BEFORE MS MA)))
               (D_AND MA MB)) (D_AND P B))
         (D_AND (D_AND PA PB) PS))
```

[hot_shared_spare]

```
⊢ ∀A B C_a C_d.
      (∀s. ALL_DISTINCT [A s; B s; C_a s]) ∧ (C_a = C_d) ⇒
      (D_AND (shared_spare A B C_a C_d)
         (shared_spare B A C_a C_d) =
       D_AND (D_AND A B) C_a)
```

[IN_REST]

```
⊢ ∀x s. x ∈ REST s ⟺ x ∈ s ∧ x ≠ CHOICE s
```

[IN_UNIONL]

```
⊢ ∀l v. v ∈ UNIONL l ⟺ ∃s. MEM s l ∧ v ∈ s
```

[indep_vars_indep_var_MA_MS]

```
⊢ ∀CS SS MA MS MB P B PA PB PS_a p.
      indep_vars_10 p lborel CS SS MA MS MB P B PA PB PS_a ⇒
      indep_var p lborel MS lborel MA
```

[indep_vars_sets_def]

```
⊢ indep_vars_sets [A0; A1; A2; A3; A4; A5; A6; A7; A8; A9] p t ⟺
  indep_vars_10 p lborel (λs. real (A0 s)) (λs. real (A1 s))
    (λs. real (A2 s)) (λs. real (A3 s)) (λs. real (A4 s))
    (λs. real (A5 s)) (λs. real (A6 s)) (λs. real (A7 s))
    (λs. real (A8 s)) (λs. real (A9 s)) ∧
  indep_sets p
    (λi.
         {if i = 0 then
            PREIMAGE (λs. real (A0 s)) {u | u ≤ t} ∩ p_space p
          else if i = 1 then
            PREIMAGE (λs. real (A1 s)) {u | u ≤ t} ∩ p_space p
          else if i = 2 then
            PREIMAGE (λs. (real (A3 s),real (A2 s)))
              {(u,w) | u < w ∧ 0 ≤ w ∧ w ≤ t} ∩ p_space p
          else if i = 3 then
            PREIMAGE (λs. real (A4 s)) {u | u ≤ t} ∩ p_space p
          else if i = 4 then
            PREIMAGE (λs. real (A5 s)) {u | u ≤ t} ∩ p_space p
          else if i = 5 then
            PREIMAGE (λs. real (A6 s)) {u | u ≤ t} ∩ p_space p
          else if i = 6 then
            PREIMAGE (λs. real (A7 s)) {u | u ≤ t} ∩ p_space p
          else if i = 7 then
            PREIMAGE (λs. real (A8 s)) {u | u ≤ t} ∩ p_space p
          else
            PREIMAGE (λs. real (A9 s)) {u | u ≤ t} ∩ p_space p})
    {0; 1; 2; 3; 4; 5; 6; 7; 8}
```

[indep_vars_sets_ind]

```
⊢ ∀P.
      (∀A0 A1 A2 A3 A4 A5 A6 A7 A8 A9 p t.
           P [A0; A1; A2; A3; A4; A5; A6; A7; A8; A9] p t) ∧
      (∀v4. P [] (FST v4) (SND v4)) ∧
      (∀v7 v8. P [v7] (FST v8) (SND v8)) ∧
      (∀v12 v11 v13. P [v12; v11] (FST v13) (SND v13)) ∧
      (∀v18 v17 v16 v19. P [v18; v17; v16] (FST v19) (SND v19)) ∧
      (∀v25 v24 v23 v22 v26.
           P [v25; v24; v23; v22] (FST v26) (SND v26)) ∧
      (∀v33 v32 v31 v30 v29 v34.
           P [v33; v32; v31; v30; v29] (FST v34) (SND v34)) ∧
      (∀v42 v41 v40 v39 v38 v37 v43.
           P [v42; v41; v40; v39; v38; v37] (FST v43) (SND v43)) ∧
      (∀v52 v51 v50 v49 v48 v47 v46 v53.
           P [v52; v51; v50; v49; v48; v47; v46] (FST v53)
             (SND v53)) ∧
      (∀v63 v62 v61 v60 v59 v58 v57 v56 v64.
           P [v63; v62; v61; v60; v59; v58; v57; v56] (FST v64)
             (SND v64)) ∧
```

$(\forall v_{75}\ v_{74}\ v_{73}\ v_{72}\ v_{71}\ v_{70}\ v_{69}\ v_{68}\ v_{67}\ v_{76}.$
$\quad P\ [v_{75};\ v_{74};\ v_{73};\ v_{72};\ v_{71};\ v_{70};\ v_{69};\ v_{68};\ v_{67}]$
$\quad\ (\texttt{FST}\ v_{76})\ (\texttt{SND}\ v_{76}))\ \land$
$(\forall v_{92}\ v_{91}\ v_{90}\ v_{89}\ v_{88}\ v_{87}\ v_{86}\ v_{85}\ v_{84}\ v_{83}\ v_{79}\ v_{80}\ v_{93}.$
$\quad P$
$\quad\ (v_{92}::v_{91}::v_{90}::v_{89}::v_{88}::v_{87}::v_{86}::v_{85}::v_{84}::$
$\quad\quad v_{83}::v_{79}::v_{80})\ (\texttt{FST}\ v_{93})\ (\texttt{SND}\ v_{93}))\ \Rightarrow$
$\forall v\ v_1\ v_2.\ P\ v\ v_1\ v_2$

[lemma_add_1]

$\vdash \forall A_{57}\ A_{58}\ A_{59}\ A_{60}\ A_{61}\ A_{62}\ A_{63}.$
$\quad A_{57} \neq \texttt{PosInf} \land A_{57} \neq \texttt{NegInf} \land A_{58} \neq \texttt{PosInf} \land$
$\quad A_{58} \neq \texttt{NegInf} \land A_{59} \neq \texttt{PosInf} \land A_{59} \neq \texttt{NegInf} \land$
$\quad A_{60} \neq \texttt{PosInf} \land A_{60} \neq \texttt{NegInf} \land A_{61} \neq \texttt{PosInf} \land$
$\quad A_{61} \neq \texttt{NegInf} \land A_{62} \neq \texttt{PosInf} \land A_{62} \neq \texttt{NegInf} \land$
$\quad A_{63} \neq \texttt{PosInf} \land A_{63} \neq \texttt{NegInf} \Rightarrow$
$\quad (A_{57} + A_{58} + A_{59} + A_{60} + A_{61} + A_{62} + -A_{63} =$
$\quad\ A_{57} + A_{58} + A_{59} + A_{60} + A_{61} + A_{62} - A_{63})$

[neg_sub]

$\vdash \forall x\ y.$
$\quad x \neq \texttt{NegInf} \land x \neq \texttt{PosInf} \lor y \neq \texttt{NegInf} \land y \neq \texttt{PosInf} \Rightarrow$
$\quad (-(x - y) = y - x)$

[normal_real_mul_10]

$\vdash \forall x_1\ x_2\ x_3\ x_4\ x_5\ x_6\ x_7\ x_8\ x_9\ x_{10}.$
$\quad x_1 \neq \texttt{PosInf} \land x_1 \neq \texttt{NegInf} \land x_2 \neq \texttt{PosInf} \land x_2 \neq \texttt{NegInf} \land$
$\quad x_3 \neq \texttt{PosInf} \land x_3 \neq \texttt{NegInf} \land x_4 \neq \texttt{PosInf} \land x_4 \neq \texttt{NegInf} \land$
$\quad x_5 \neq \texttt{PosInf} \land x_5 \neq \texttt{NegInf} \land x_6 \neq \texttt{PosInf} \land x_6 \neq \texttt{NegInf} \land$
$\quad x_7 \neq \texttt{PosInf} \land x_7 \neq \texttt{NegInf} \land x_8 \neq \texttt{PosInf} \land x_8 \neq \texttt{NegInf} \land$
$\quad x_9 \neq \texttt{PosInf} \land x_9 \neq \texttt{NegInf} \land x_{10} \neq \texttt{PosInf} \land x_{10} \neq \texttt{NegInf} \Rightarrow$
$\quad (\texttt{Normal}$
$\quad\quad (\texttt{real}\ x_1 \times \texttt{real}\ x_2 \times \texttt{real}\ x_3 \times \texttt{real}\ x_4 \times \texttt{real}\ x_5 \times$
$\quad\quad\ \texttt{real}\ x_6 \times \texttt{real}\ x_7 \times \texttt{real}\ x_8 \times \texttt{real}\ x_9 \times \texttt{real}\ x_{10}) =$
$\quad\ \texttt{Normal}\ (\texttt{real}\ x_1) \times \texttt{Normal}\ (\texttt{real}\ x_2) \times \texttt{Normal}\ (\texttt{real}\ x_3) \times$
$\quad\ \texttt{Normal}\ (\texttt{real}\ x_4) \times \texttt{Normal}\ (\texttt{real}\ x_5) \times \texttt{Normal}\ (\texttt{real}\ x_6) \times$
$\quad\ \texttt{Normal}\ (\texttt{real}\ x_7) \times \texttt{Normal}\ (\texttt{real}\ x_8) \times \texttt{Normal}\ (\texttt{real}\ x_9) \times$
$\quad\ \texttt{Normal}\ (\texttt{real}\ x_{10}))$

[normal_real_mul_3]

$\vdash \forall x\ y\ z.$
$\quad x \neq \texttt{PosInf} \land x \neq \texttt{NegInf} \land y \neq \texttt{PosInf} \land y \neq \texttt{NegInf} \land$
$\quad z \neq \texttt{PosInf} \land z \neq \texttt{NegInf} \Rightarrow$
$\quad (\texttt{Normal}\ (\texttt{real}\ x \times \texttt{real}\ y \times \texttt{real}\ z) =$
$\quad\ \texttt{Normal}\ (\texttt{real}\ x) \times \texttt{Normal}\ (\texttt{real}\ y) \times \texttt{Normal}\ (\texttt{real}\ z))$

[normal_real_mul_4]

⊢ $\forall x_1\ x_2\ x_3\ x_4$.
$x_1 \neq$ PosInf ∧ $x_1 \neq$ NegInf ∧ $x_2 \neq$ PosInf ∧ $x_2 \neq$ NegInf ∧
$x_3 \neq$ PosInf ∧ $x_3 \neq$ NegInf ∧ $x_4 \neq$ PosInf ∧ $x_4 \neq$ NegInf ⇒
(Normal (real $x_1$ × real $x_2$ × real $x_3$ × real $x_4$) =
Normal (real $x_1$) × Normal (real $x_2$) × Normal (real $x_3$) ×
Normal (real $x_4$))

[normal_real_mul_5]

⊢ $\forall x_1\ x_2\ x_3\ x_4\ x_5$.
$x_1 \neq$ PosInf ∧ $x_1 \neq$ NegInf ∧ $x_2 \neq$ PosInf ∧ $x_2 \neq$ NegInf ∧
$x_3 \neq$ PosInf ∧ $x_3 \neq$ NegInf ∧ $x_4 \neq$ PosInf ∧ $x_4 \neq$ NegInf ∧
$x_5 \neq$ PosInf ∧ $x_5 \neq$ NegInf ⇒
(Normal (real $x_1$ × real $x_2$ × real $x_3$ × real $x_4$ × real $x_5$) =
Normal (real $x_1$) × Normal (real $x_2$) × Normal (real $x_3$) ×
Normal (real $x_4$) × Normal (real $x_5$))

[normal_real_mul_6]

⊢ $\forall x_1\ x_2\ x_3\ x_4\ x_5\ x_6$.
$x_1 \neq$ PosInf ∧ $x_1 \neq$ NegInf ∧ $x_2 \neq$ PosInf ∧ $x_2 \neq$ NegInf ∧
$x_3 \neq$ PosInf ∧ $x_3 \neq$ NegInf ∧ $x_4 \neq$ PosInf ∧ $x_4 \neq$ NegInf ∧
$x_5 \neq$ PosInf ∧ $x_5 \neq$ NegInf ∧ $x_6 \neq$ PosInf ∧ $x_6 \neq$ NegInf ⇒
(Normal
(real $x_1$ × real $x_2$ × real $x_3$ × real $x_4$ × real $x_5$ ×
real $x_6$) =
Normal (real $x_1$) × Normal (real $x_2$) × Normal (real $x_3$) ×
Normal (real $x_4$) × Normal (real $x_5$) × Normal (real $x_6$))

[normal_real_mul_7]

⊢ $\forall x_1\ x_2\ x_3\ x_4\ x_5\ x_6\ x_7$.
$x_1 \neq$ PosInf ∧ $x_1 \neq$ NegInf ∧ $x_2 \neq$ PosInf ∧ $x_2 \neq$ NegInf ∧
$x_3 \neq$ PosInf ∧ $x_3 \neq$ NegInf ∧ $x_4 \neq$ PosInf ∧ $x_4 \neq$ NegInf ∧
$x_5 \neq$ PosInf ∧ $x_5 \neq$ NegInf ∧ $x_6 \neq$ PosInf ∧ $x_6 \neq$ NegInf ∧
$x_7 \neq$ PosInf ∧ $x_7 \neq$ NegInf ⇒
(Normal
(real $x_1$ × real $x_2$ × real $x_3$ × real $x_4$ × real $x_5$ ×
real $x_6$ × real $x_7$) =
Normal (real $x_1$) × Normal (real $x_2$) × Normal (real $x_3$) ×
Normal (real $x_4$) × Normal (real $x_5$) × Normal (real $x_6$) ×
Normal (real $x_7$))

[normal_real_mul_8]

⊢ $\forall x_1\ x_2\ x_3\ x_4\ x_5\ x_6\ x_7\ x_8$.
$x_1 \neq$ PosInf ∧ $x_1 \neq$ NegInf ∧ $x_2 \neq$ PosInf ∧ $x_2 \neq$ NegInf ∧
$x_3 \neq$ PosInf ∧ $x_3 \neq$ NegInf ∧ $x_4 \neq$ PosInf ∧ $x_4 \neq$ NegInf ∧
$x_5 \neq$ PosInf ∧ $x_5 \neq$ NegInf ∧ $x_6 \neq$ PosInf ∧ $x_6 \neq$ NegInf ∧
$x_7 \neq$ PosInf ∧ $x_7 \neq$ NegInf ∧ $x_8 \neq$ PosInf ∧ $x_8 \neq$ NegInf ⇒
(Normal

```
      (real x1 × real x2 × real x3 × real x4 × real x5 ×
       real x6 × real x7 × real x8) =
     Normal (real x1) × Normal (real x2) × Normal (real x3) ×
     Normal (real x4) × Normal (real x5) × Normal (real x6) ×
     Normal (real x7) × Normal (real x8))
```

[normal_real_mul_9]

```
⊢ ∀x1 x2 x3 x4 x5 x6 x7 x8 x9.
    x1 ≠ PosInf ∧ x1 ≠ NegInf ∧ x2 ≠ PosInf ∧ x2 ≠ NegInf ∧
    x3 ≠ PosInf ∧ x3 ≠ NegInf ∧ x4 ≠ PosInf ∧ x4 ≠ NegInf ∧
    x5 ≠ PosInf ∧ x5 ≠ NegInf ∧ x6 ≠ PosInf ∧ x6 ≠ NegInf ∧
    x7 ≠ PosInf ∧ x7 ≠ NegInf ∧ x8 ≠ PosInf ∧ x8 ≠ NegInf ∧
    x9 ≠ PosInf ∧ x9 ≠ NegInf ⇒
    (Normal
      (real x1 × real x2 × real x3 × real x4 × real x5 ×
       real x6 × real x7 × real x8 × real x9) =
     Normal (real x1) × Normal (real x2) × Normal (real x3) ×
     Normal (real x4) × Normal (real x5) × Normal (real x6) ×
     Normal (real x7) × Normal (real x8) × Normal (real x9))
```

[OR_AND_ID_LEFT]

```
⊢ ∀A B. D_OR A (D_AND A B) = A
```

[PIE_lem4_63_prod]

```
⊢ ∀f A B C D E k.
    ALL_DISTINCT [A; B; C; D; E; k] ∧
    (∀x.
       x ∈
       {{A}; {B}; {C}; {D}; {E}; {k}; {A; B}; {A; C};
        {A; D}; {A; E}; {A; k}; {B; C}; {B; D}; {B; E};
        {B; k}; {C; D}; {C; E}; {C; k}; {D; E}; {D; k};
        {E; k}; {A; B; C}; {A; B; D}; {A; B; E}; {A; B; k};
        {A; C; D}; {A; C; E}; {A; C; k}; {A; D; E};
        {A; D; k}; {A; E; k}; {B; C; D}; {B; C; E};
        {B; C; k}; {B; D; E}; {B; D; k}; {B; E; k};
        {C; D; E}; {C; D; k}; {C; E; k}; {D; E; k};
        {A; B; C; D}; {A; B; C; E}; {A; B; C; k};
        {A; B; D; E}; {A; B; D; k}; {A; B; E; k};
        {A; C; D; E}; {A; C; D; k}; {A; C; E; k};
        {A; D; E; k}; {B; C; D; E}; {B; C; D; k};
        {B; C; E; k}; {B; D; E; k}; {C; D; E; k};
        {A; B; C; D; E}; {A; B; C; D; k}; {A; B; C; E; k};
        {A; B; D; E; k}; {A; C; D; E; k}; {B; C; D; E; k};
        {A; B; C; D; E; k}} ⇒
       f x ≠ PosInf) ⇒
    (SIGMA f
      {{A}; {B}; {C}; {D}; {E}; {k}; {A; B}; {A; C}; {A; D};
       {A; E}; {A; k}; {B; C}; {B; D}; {B; E}; {B; k};
```

{C; D}; {C; E}; {C; k}; {D; E}; {D; k}; {E; k};
{A; B; C}; {A; B; D}; {A; B; E}; {A; B; k};
{A; C; D}; {A; C; E}; {A; C; k}; {A; D; E};
{A; D; k}; {A; E; k}; {B; C; D}; {B; C; E};
{B; C; k}; {B; D; E}; {B; D; k}; {B; E; k};
{C; D; E}; {C; D; k}; {C; E; k}; {D; E; k};
{A; B; C; D}; {A; B; C; E}; {A; B; C; k};
{A; B; D; E}; {A; B; D; k}; {A; B; E; k};
{A; C; D; E}; {A; C; D; k}; {A; C; E; k};
{A; D; E; k}; {B; C; D; E}; {B; C; D; k};
{B; C; E; k}; {B; D; E; k}; {C; D; E; k};
{A; B; C; D; E}; {A; B; C; D; k}; {A; B; C; E; k};
{A; B; D; E; k}; {A; C; D; E; k}; {B; C; D; E; k};
{A; B; C; D; E; k}} =
f {A} + f {B} + f {C} + f {D} + f {E} + f {k} +
f {A; B} + f {A; C} + f {A; D} + f {A; E} + f {A; k} +
f {B; C} + f {B; D} + f {B; E} + f {B; k} + f {C; D} +
f {C; E} + f {C; k} + f {D; E} + f {D; k} + f {E; k} +
f {A; B; C} + f {A; B; D} + f {A; B; E} + f {A; B; k} +
f {A; C; D} + f {A; C; E} + f {A; C; k} + f {A; D; E} +
f {A; D; k} + f {A; E; k} + f {B; C; D} + f {B; C; E} +
f {B; C; k} + f {B; D; E} + f {B; D; k} + f {B; E; k} +
f {C; D; E} + f {C; D; k} + f {C; E; k} + f {D; E; k} +
f {A; B; C; D} + f {A; B; C; E} + f {A; B; C; k} +
f {A; B; D; E} + f {A; B; D; k} + f {A; B; E; k} +
f {A; C; D; E} + f {A; C; D; k} + f {A; C; E; k} +
f {A; D; E; k} + f {B; C; D; E} + f {B; C; D; k} +
f {B; C; E; k} + f {B; D; E; k} + f {C; D; E; k} +
f {A; B; C; D; E} + f {A; B; C; D; k} +
f {A; B; C; E; k} + f {A; B; D; E; k} +
f {A; C; D; E; k} + f {B; C; D; E; k} +
f {A; B; C; D; E; k})

[PIE_set_alt_form_lem_6]

⊢ ∀A B C D E k.
{t | t ⊆ {A; B; C; D; E; k} ∧ t ≠ {}} =
{{A}; {B}; {C}; {D}; {E}; {k}; {A; B}; {A; C}; {A; D};
{A; E}; {A; k}; {B; C}; {B; D}; {B; E}; {B; k}; {C; D};
{C; E}; {C; k}; {D; E}; {D; k}; {E; k}; {A; B; C};
{A; B; D}; {A; B; E}; {A; B; k}; {A; C; D}; {A; C; E};
{A; C; k}; {A; D; E}; {A; D; k}; {A; E; k}; {B; C; D};
{B; C; E}; {B; C; k}; {B; D; E}; {B; D; k}; {B; E; k};
{C; D; E}; {C; D; k}; {C; E; k}; {D; E; k};
{A; B; C; D}; {A; B; C; E}; {A; B; C; k}; {A; B; D; E};
{A; B; D; k}; {A; B; E; k}; {A; C; D; E}; {A; C; D; k};
{A; C; E; k}; {A; D; E; k}; {B; C; D; E}; {B; C; D; k};
{B; C; E; k}; {B; D; E; k}; {C; D; E; k};
{A; B; C; D; E}; {A; B; C; D; k}; {A; B; C; E; k};
{A; B; D; E; k}; {A; C; D; E; k}; {B; C; D; E; k};

$\{A;\ B;\ C;\ D;\ E;\ k\}\}$

[POW_DELETE_EMPTY]

$\vdash \forall s.$ POW $s$ DELETE $\{\}$ = $\{t \mid t \subseteq s \land t \neq \{\}\}$

[POW_set_2]

$\vdash \forall A_1\ A_2.$ POW $\{A_1;\ A_2\}$ = $\{\{\};\ \{A_1\};\ \{A_2\};\ \{A_1;\ A_2\}\}$

[POW_set_3]

$\vdash \forall A_1\ A_2\ A_3.$
POW $\{A_1;\ A_2;\ A_3\}$ =
$\{\{\};\ \{A_1\};\ \{A_2\};\ \{A_3\};\ \{A_1;\ A_2\};\ \{A_1;\ A_3\};\ \{A_2;\ A_3\};$
$\{A_1;\ A_2;\ A_3\}\}$

[POW_set_4]

$\vdash \forall A\ B\ C\ D.$
POW $\{A;\ B;\ C;\ D\}$ =
$\{\{\};\ \{A\};\ \{B\};\ \{C\};\ \{D\};\ \{A;\ B\};\ \{A;\ C\};\ \{A;\ D\};\ \{B;\ C\};$
$\{B;\ D\};\ \{C;\ D\};\ \{A;\ B;\ C\};\ \{A;\ B;\ D\};\ \{A;\ C;\ D\};$
$\{B;\ C;\ D\};\ \{A;\ B;\ C;\ D\}\}$

[POW_set_5]

$\vdash \forall A\ B\ C\ D\ E.$
POW $\{A;\ B;\ C;\ D;\ E\}$ =
$\{\{\};\ \{A\};\ \{B\};\ \{C\};\ \{D\};\ \{E\};\ \{A;\ B\};\ \{A;\ C\};\ \{A;\ D\};$
$\{A;\ E\};\ \{B;\ C\};\ \{B;\ D\};\ \{B;\ E\};\ \{C;\ D\};\ \{C;\ E\};\ \{D;\ E\};$
$\{A;\ B;\ C\};\ \{A;\ B;\ D\};\ \{A;\ B;\ E\};\ \{A;\ C;\ D\};\ \{A;\ C;\ E\};$
$\{A;\ D;\ E\};\ \{B;\ C;\ D\};\ \{B;\ C;\ E\};\ \{B;\ D;\ E\};\ \{C;\ D;\ E\};$
$\{A;\ B;\ C;\ D\};\ \{A;\ B;\ C;\ E\};\ \{A;\ B;\ D;\ E\};\ \{A;\ C;\ D;\ E\};$
$\{B;\ C;\ D;\ E\};\ \{A;\ B;\ C;\ D;\ E\}\}$

[POW_set_6]

$\vdash \forall A\ B\ C\ D\ E\ k.$
POW $\{A;\ B;\ C;\ D;\ E;\ k\}$ =
$\{\{\};\ \{A\};\ \{B\};\ \{C\};\ \{D\};\ \{E\};\ \{k\};\ \{A;\ B\};\ \{A;\ C\};$
$\{A;\ D\};\ \{A;\ E\};\ \{A;\ k\};\ \{B;\ C\};\ \{B;\ D\};\ \{B;\ E\};\ \{B;\ k\};$
$\{C;\ D\};\ \{C;\ E\};\ \{C;\ k\};\ \{D;\ E\};\ \{D;\ k\};\ \{E;\ k\};$
$\{A;\ B;\ C\};\ \{A;\ B;\ D\};\ \{A;\ B;\ E\};\ \{A;\ B;\ k\};\ \{A;\ C;\ D\};$
$\{A;\ C;\ E\};\ \{A;\ C;\ k\};\ \{A;\ D;\ E\};\ \{A;\ D;\ k\};\ \{A;\ E;\ k\};$
$\{B;\ C;\ D\};\ \{B;\ C;\ E\};\ \{B;\ C;\ k\};\ \{B;\ D;\ E\};\ \{B;\ D;\ k\};$
$\{B;\ E;\ k\};\ \{C;\ D;\ E\};\ \{C;\ D;\ k\};\ \{C;\ E;\ k\};\ \{D;\ E;\ k\};$
$\{A;\ B;\ C;\ D\};\ \{A;\ B;\ C;\ E\};\ \{A;\ B;\ C;\ k\};\ \{A;\ B;\ D;\ E\};$
$\{A;\ B;\ D;\ k\};\ \{A;\ B;\ E;\ k\};\ \{A;\ C;\ D;\ E\};\ \{A;\ C;\ D;\ k\};$
$\{A;\ C;\ E;\ k\};\ \{A;\ D;\ E;\ k\};\ \{B;\ C;\ D;\ E\};\ \{B;\ C;\ D;\ k\};$
$\{B;\ C;\ E;\ k\};\ \{B;\ D;\ E;\ k\};\ \{C;\ D;\ E;\ k\};$
$\{A;\ B;\ C;\ D;\ E\};\ \{A;\ B;\ C;\ D;\ k\};\ \{A;\ B;\ C;\ E;\ k\};$
$\{A;\ B;\ D;\ E;\ k\};\ \{A;\ C;\ D;\ E;\ k\};\ \{B;\ C;\ D;\ E;\ k\};$
$\{A;\ B;\ C;\ D;\ E;\ k\}\}$

[POW_set_6_DELETE_EMPTY]

$\vdash \forall A\ B\ C\ D\ E\ k.$
POW $\{A;\ B;\ C;\ D;\ E;\ k\}$ DELETE $\{\}$ =
$\{\{A\};\ \{B\};\ \{C\};\ \{D\};\ \{E\};\ \{k\};\ \{A;\ B\};\ \{A;\ C\};\ \{A;\ D\};$
$\{A;\ E\};\ \{A;\ k\};\ \{B;\ C\};\ \{B;\ D\};\ \{B;\ E\};\ \{B;\ k\};\ \{C;\ D\};$
$\{C;\ E\};\ \{C;\ k\};\ \{D;\ E\};\ \{D;\ k\};\ \{E;\ k\};\ \{A;\ B;\ C\};$
$\{A;\ B;\ D\};\ \{A;\ B;\ E\};\ \{A;\ B;\ k\};\ \{A;\ C;\ D\};\ \{A;\ C;\ E\};$
$\{A;\ C;\ k\};\ \{A;\ D;\ E\};\ \{A;\ D;\ k\};\ \{A;\ E;\ k\};\ \{B;\ C;\ D\};$
$\{B;\ C;\ E\};\ \{B;\ C;\ k\};\ \{B;\ D;\ E\};\ \{B;\ D;\ k\};\ \{B;\ E;\ k\};$
$\{C;\ D;\ E\};\ \{C;\ D;\ k\};\ \{C;\ E;\ k\};\ \{D;\ E;\ k\};$
$\{A;\ B;\ C;\ D\};\ \{A;\ B;\ C;\ E\};\ \{A;\ B;\ C;\ k\};\ \{A;\ B;\ D;\ E\};$
$\{A;\ B;\ D;\ k\};\ \{A;\ B;\ E;\ k\};\ \{A;\ C;\ D;\ E\};\ \{A;\ C;\ D;\ k\};$
$\{A;\ C;\ E;\ k\};\ \{A;\ D;\ E;\ k\};\ \{B;\ C;\ D;\ E\};\ \{B;\ C;\ D;\ k\};$
$\{B;\ C;\ E;\ k\};\ \{B;\ D;\ E;\ k\};\ \{C;\ D;\ E;\ k\};$
$\{A;\ B;\ C;\ D;\ E\};\ \{A;\ B;\ C;\ D;\ k\};\ \{A;\ B;\ C;\ E;\ k\};$
$\{A;\ B;\ D;\ E;\ k\};\ \{A;\ C;\ D;\ E;\ k\};\ \{B;\ C;\ D;\ E;\ k\};$
$\{A;\ B;\ C;\ D;\ E;\ k\}\}$

[PROB_PIE_63]

$\vdash \forall A_1\ A_2\ A_3\ A_4\ A_5\ A_6\ p.$
ALL_DISTINCT $[A_1;\ A_2;\ A_3;\ A_4;\ A_5;\ A_6]$ $\wedge$ prob_space $p$ $\wedge$
$A_1 \in$ events $p$ $\wedge$ $A_2 \in$ events $p$ $\wedge$ $A_3 \in$ events $p$ $\wedge$
$A_4 \in$ events $p$ $\wedge$ $A_5 \in$ events $p$ $\wedge$ $A_6 \in$ events $p$ $\Rightarrow$
(prob $p$ (union_list $[A_1;\ A_2;\ A_3;\ A_4;\ A_5;\ A_6]$) =
prob $p$ $A_1$ + prob $p$ $A_2$ + prob $p$ $A_3$ + prob $p$ $A_4$ +
prob $p$ $A_5$ + prob $p$ $A_6$ - prob $p$ $(A_1 \cap A_2)$ -
prob $p$ $(A_1 \cap A_3)$ - prob $p$ $(A_1 \cap A_4)$ - prob $p$ $(A_1 \cap A_5)$ -
prob $p$ $(A_1 \cap A_6)$ - prob $p$ $(A_2 \cap A_3)$ - prob $p$ $(A_2 \cap A_4)$ -
prob $p$ $(A_2 \cap A_5)$ - prob $p$ $(A_2 \cap A_6)$ - prob $p$ $(A_3 \cap A_4)$ -
prob $p$ $(A_3 \cap A_5)$ - prob $p$ $(A_3 \cap A_6)$ - prob $p$ $(A_4 \cap A_5)$ -
prob $p$ $(A_4 \cap A_6)$ - prob $p$ $(A_5 \cap A_6)$ +
prob $p$ $(A_1 \cap A_2 \cap A_3)$ + prob $p$ $(A_1 \cap A_2 \cap A_4)$ +
prob $p$ $(A_1 \cap A_2 \cap A_5)$ + prob $p$ $(A_1 \cap A_2 \cap A_6)$ +
prob $p$ $(A_1 \cap A_3 \cap A_4)$ + prob $p$ $(A_1 \cap A_3 \cap A_5)$ +
prob $p$ $(A_1 \cap A_3 \cap A_6)$ + prob $p$ $(A_1 \cap A_4 \cap A_5)$ +
prob $p$ $(A_1 \cap A_4 \cap A_6)$ + prob $p$ $(A_1 \cap A_5 \cap A_6)$ +
prob $p$ $(A_2 \cap A_3 \cap A_4)$ + prob $p$ $(A_2 \cap A_3 \cap A_5)$ +
prob $p$ $(A_2 \cap A_3 \cap A_6)$ + prob $p$ $(A_2 \cap A_4 \cap A_5)$ +
prob $p$ $(A_2 \cap A_4 \cap A_6)$ + prob $p$ $(A_2 \cap A_5 \cap A_6)$ +
prob $p$ $(A_3 \cap A_4 \cap A_5)$ + prob $p$ $(A_3 \cap A_4 \cap A_6)$ +
prob $p$ $(A_3 \cap A_5 \cap A_6)$ + prob $p$ $(A_4 \cap A_5 \cap A_6)$ -
prob $p$ $(A_1 \cap A_2 \cap A_3 \cap A_4)$ -
prob $p$ $(A_1 \cap A_2 \cap A_3 \cap A_5)$ -
prob $p$ $(A_1 \cap A_2 \cap A_3 \cap A_6)$ -
prob $p$ $(A_1 \cap A_2 \cap A_4 \cap A_5)$ -
prob $p$ $(A_1 \cap A_2 \cap A_4 \cap A_6)$ -
prob $p$ $(A_1 \cap A_2 \cap A_5 \cap A_6)$ -
prob $p$ $(A_1 \cap A_3 \cap A_4 \cap A_5)$ -
prob $p$ $(A_1 \cap A_3 \cap A_4 \cap A_6)$ -

```
        prob p (A_1 ∩ A_3 ∩ A_5 ∩ A_6) -
        prob p (A_1 ∩ A_4 ∩ A_5 ∩ A_6) -
        prob p (A_2 ∩ A_3 ∩ A_4 ∩ A_5) -
        prob p (A_2 ∩ A_3 ∩ A_4 ∩ A_6) -
        prob p (A_2 ∩ A_3 ∩ A_5 ∩ A_6) -
        prob p (A_2 ∩ A_4 ∩ A_5 ∩ A_6) -
        prob p (A_3 ∩ A_4 ∩ A_5 ∩ A_6) +
        prob p (A_1 ∩ A_2 ∩ A_3 ∩ A_4 ∩ A_5) +
        prob p (A_1 ∩ A_2 ∩ A_3 ∩ A_4 ∩ A_6) +
        prob p (A_1 ∩ A_2 ∩ A_3 ∩ A_5 ∩ A_6) +
        prob p (A_1 ∩ A_2 ∩ A_4 ∩ A_5 ∩ A_6) +
        prob p (A_1 ∩ A_3 ∩ A_4 ∩ A_5 ∩ A_6) +
        prob p (A_2 ∩ A_3 ∩ A_4 ∩ A_5 ∩ A_6) -
        prob p (A_1 ∩ A_2 ∩ A_3 ∩ A_4 ∩ A_5 ∩ A_6))
```

[PROB_PIE_CAS_63_lem1]

```
 ⊢ ∀ CS SS MA MS MB P B PA PB PS_a p t.
       ALL_DISTINCT
         [DFT_event p CS t; DFT_event p SS t;
          DFT_event p (D_AND MA (D_BEFORE MS MA)) t;
          DFT_event p (D_AND MA MB) t;
          DFT_event p (D_AND P B) t;
          DFT_event p (D_AND (D_AND PA PB) PS_a) t] ∧
       prob_space p ∧ DFT_event p CS t ∈ events p ∧
       DFT_event p SS t ∈ events p ∧
       DFT_event p (D_AND MA (D_BEFORE MS MA)) t ∈ events p ∧
       DFT_event p (D_AND MA MB) t ∈ events p ∧
       DFT_event p (D_AND P B) t ∈ events p ∧
       DFT_event p (D_AND (D_AND PA PB) PS_a) t ∈ events p ⇒
       (prob p
          (union_list
             [DFT_event p CS t; DFT_event p SS t;
              DFT_event p (D_AND MA (D_BEFORE MS MA)) t;
              DFT_event p (D_AND MA MB) t;
              DFT_event p (D_AND P B) t;
              DFT_event p (D_AND (D_AND PA PB) PS_a) t]) =
        prob p (DFT_event p CS t) + prob p (DFT_event p SS t) +
        prob p (DFT_event p (D_AND MA (D_BEFORE MS MA)) t) +
        prob p (DFT_event p (D_AND MA MB) t) +
        prob p (DFT_event p (D_AND P B) t) +
        prob p (DFT_event p (D_AND (D_AND PA PB) PS_a) t) -
        prob p (DFT_event p CS t ∩ DFT_event p SS t) -
        prob p
          (DFT_event p CS t ∩
           DFT_event p (D_AND MA (D_BEFORE MS MA)) t) -
        prob p (DFT_event p CS t ∩ DFT_event p (D_AND MA MB) t) -
        prob p (DFT_event p CS t ∩ DFT_event p (D_AND P B) t) -
        prob p
          (DFT_event p CS t ∩
```

```
   DFT_event p (D_AND (D_AND PA PB) PS_a) t) -
prob p
  (DFT_event p SS t ∩
   DFT_event p (D_AND MA (D_BEFORE MS MA)) t) -
prob p (DFT_event p SS t ∩ DFT_event p (D_AND MA MB) t) -
prob p (DFT_event p SS t ∩ DFT_event p (D_AND P B) t) -
prob p
  (DFT_event p SS t ∩
   DFT_event p (D_AND (D_AND PA PB) PS_a) t) -
prob p
  (DFT_event p (D_AND MA (D_BEFORE MS MA)) t ∩
   DFT_event p (D_AND MA MB) t) -
prob p
  (DFT_event p (D_AND MA (D_BEFORE MS MA)) t ∩
   DFT_event p (D_AND P B) t) -
prob p
  (DFT_event p (D_AND MA (D_BEFORE MS MA)) t ∩
   DFT_event p (D_AND (D_AND PA PB) PS_a) t) -
prob p
  (DFT_event p (D_AND MA MB) t ∩
   DFT_event p (D_AND P B) t) -
prob p
  (DFT_event p (D_AND MA MB) t ∩
   DFT_event p (D_AND (D_AND PA PB) PS_a) t) -
prob p
  (DFT_event p (D_AND P B) t ∩
   DFT_event p (D_AND (D_AND PA PB) PS_a) t) +
prob p
  (DFT_event p CS t ∩ DFT_event p SS t ∩
   DFT_event p (D_AND MA (D_BEFORE MS MA)) t) +
prob p
  (DFT_event p CS t ∩ DFT_event p SS t ∩
   DFT_event p (D_AND MA MB) t) +
prob p
  (DFT_event p CS t ∩ DFT_event p SS t ∩
   DFT_event p (D_AND P B) t) +
prob p
  (DFT_event p CS t ∩ DFT_event p SS t ∩
   DFT_event p (D_AND (D_AND PA PB) PS_a) t) +
prob p
  (DFT_event p CS t ∩
   DFT_event p (D_AND MA (D_BEFORE MS MA)) t ∩
   DFT_event p (D_AND MA MB) t) +
prob p
  (DFT_event p CS t ∩
   DFT_event p (D_AND MA (D_BEFORE MS MA)) t ∩
   DFT_event p (D_AND P B) t) +
prob p
  (DFT_event p CS t ∩
```

```
    DFT_event p (D_AND MA (D_BEFORE MS MA)) t ∩
    DFT_event p (D_AND (D_AND PA PB) PS_a) t) +
prob p
  (DFT_event p CS t ∩ DFT_event p (D_AND MA MB) t ∩
    DFT_event p (D_AND P B) t) +
prob p
  (DFT_event p CS t ∩ DFT_event p (D_AND MA MB) t ∩
    DFT_event p (D_AND (D_AND PA PB) PS_a) t) +
prob p
  (DFT_event p CS t ∩ DFT_event p (D_AND P B) t ∩
    DFT_event p (D_AND (D_AND PA PB) PS_a) t) +
prob p
  (DFT_event p SS t ∩
    DFT_event p (D_AND MA (D_BEFORE MS MA)) t ∩
    DFT_event p (D_AND MA MB) t) +
prob p
  (DFT_event p SS t ∩
    DFT_event p (D_AND MA (D_BEFORE MS MA)) t ∩
    DFT_event p (D_AND P B) t) +
prob p
  (DFT_event p SS t ∩
    DFT_event p (D_AND MA (D_BEFORE MS MA)) t ∩
    DFT_event p (D_AND (D_AND PA PB) PS_a) t) +
prob p
  (DFT_event p SS t ∩ DFT_event p (D_AND MA MB) t ∩
    DFT_event p (D_AND P B) t) +
prob p
  (DFT_event p SS t ∩ DFT_event p (D_AND MA MB) t ∩
    DFT_event p (D_AND (D_AND PA PB) PS_a) t) +
prob p
  (DFT_event p SS t ∩ DFT_event p (D_AND P B) t ∩
    DFT_event p (D_AND (D_AND PA PB) PS_a) t) +
prob p
  (DFT_event p (D_AND MA (D_BEFORE MS MA)) t ∩
    DFT_event p (D_AND MA MB) t ∩
    DFT_event p (D_AND P B) t) +
prob p
  (DFT_event p (D_AND MA (D_BEFORE MS MA)) t ∩
    DFT_event p (D_AND MA MB) t ∩
    DFT_event p (D_AND (D_AND PA PB) PS_a) t) +
prob p
  (DFT_event p (D_AND MA (D_BEFORE MS MA)) t ∩
    DFT_event p (D_AND P B) t ∩
    DFT_event p (D_AND (D_AND PA PB) PS_a) t) +
prob p
  (DFT_event p (D_AND MA MB) t ∩
    DFT_event p (D_AND P B) t ∩
    DFT_event p (D_AND (D_AND PA PB) PS_a) t) -
prob p
```

```
  (DFT_event p CS t ∩ DFT_event p SS t ∩
   DFT_event p (D_AND MA (D_BEFORE MS MA)) t ∩
   DFT_event p (D_AND MA MB) t) -
prob p
  (DFT_event p CS t ∩ DFT_event p SS t ∩
   DFT_event p (D_AND MA (D_BEFORE MS MA)) t ∩
   DFT_event p (D_AND P B) t) -
prob p
  (DFT_event p CS t ∩ DFT_event p SS t ∩
   DFT_event p (D_AND MA (D_BEFORE MS MA)) t ∩
   DFT_event p (D_AND (D_AND PA PB) PS_a) t) -
prob p
  (DFT_event p CS t ∩ DFT_event p SS t ∩
   DFT_event p (D_AND MA MB) t ∩
   DFT_event p (D_AND P B) t) -
prob p
  (DFT_event p CS t ∩ DFT_event p SS t ∩
   DFT_event p (D_AND MA MB) t ∩
   DFT_event p (D_AND (D_AND PA PB) PS_a) t) -
prob p
  (DFT_event p CS t ∩ DFT_event p SS t ∩
   DFT_event p (D_AND P B) t ∩
   DFT_event p (D_AND (D_AND PA PB) PS_a) t) -
prob p
  (DFT_event p CS t ∩
   DFT_event p (D_AND MA (D_BEFORE MS MA)) t ∩
   DFT_event p (D_AND MA MB) t ∩
   DFT_event p (D_AND P B) t) -
prob p
  (DFT_event p CS t ∩
   DFT_event p (D_AND MA (D_BEFORE MS MA)) t ∩
   DFT_event p (D_AND MA MB) t ∩
   DFT_event p (D_AND (D_AND PA PB) PS_a) t) -
prob p
  (DFT_event p CS t ∩
   DFT_event p (D_AND MA (D_BEFORE MS MA)) t ∩
   DFT_event p (D_AND P B) t ∩
   DFT_event p (D_AND (D_AND PA PB) PS_a) t) -
prob p
  (DFT_event p CS t ∩ DFT_event p (D_AND MA MB) t ∩
   DFT_event p (D_AND P B) t ∩
   DFT_event p (D_AND (D_AND PA PB) PS_a) t) -
prob p
  (DFT_event p SS t ∩
   DFT_event p (D_AND MA (D_BEFORE MS MA)) t ∩
   DFT_event p (D_AND MA MB) t ∩
   DFT_event p (D_AND P B) t) -
prob p
  (DFT_event p SS t ∩
```

```
    DFT_event p (D_AND MA (D_BEFORE MS MA)) t ∩
    DFT_event p (D_AND MA MB) t ∩
    DFT_event p (D_AND (D_AND PA PB) PS_a) t) -
prob p
  (DFT_event p SS t ∩
    DFT_event p (D_AND MA (D_BEFORE MS MA)) t ∩
    DFT_event p (D_AND P B) t ∩
    DFT_event p (D_AND (D_AND PA PB) PS_a) t) -
prob p
  (DFT_event p SS t ∩ DFT_event p (D_AND MA MB) t ∩
    DFT_event p (D_AND P B) t ∩
    DFT_event p (D_AND (D_AND PA PB) PS_a) t) -
prob p
  (DFT_event p (D_AND MA (D_BEFORE MS MA)) t ∩
    DFT_event p (D_AND MA MB) t ∩
    DFT_event p (D_AND P B) t ∩
    DFT_event p (D_AND (D_AND PA PB) PS_a) t) +
prob p
  (DFT_event p CS t ∩ DFT_event p SS t ∩
    DFT_event p (D_AND MA (D_BEFORE MS MA)) t ∩
    DFT_event p (D_AND MA MB) t ∩
    DFT_event p (D_AND P B) t) +
prob p
  (DFT_event p CS t ∩ DFT_event p SS t ∩
    DFT_event p (D_AND MA (D_BEFORE MS MA)) t ∩
    DFT_event p (D_AND MA MB) t ∩
    DFT_event p (D_AND (D_AND PA PB) PS_a) t) +
prob p
  (DFT_event p CS t ∩ DFT_event p SS t ∩
    DFT_event p (D_AND MA (D_BEFORE MS MA)) t ∩
    DFT_event p (D_AND P B) t ∩
    DFT_event p (D_AND (D_AND PA PB) PS_a) t) +
prob p
  (DFT_event p CS t ∩ DFT_event p SS t ∩
    DFT_event p (D_AND MA MB) t ∩
    DFT_event p (D_AND P B) t ∩
    DFT_event p (D_AND (D_AND PA PB) PS_a) t) +
prob p
  (DFT_event p CS t ∩
    DFT_event p (D_AND MA (D_BEFORE MS MA)) t ∩
    DFT_event p (D_AND MA MB) t ∩
    DFT_event p (D_AND P B) t ∩
    DFT_event p (D_AND (D_AND PA PB) PS_a) t) +
prob p
  (DFT_event p SS t ∩
    DFT_event p (D_AND MA (D_BEFORE MS MA)) t ∩
    DFT_event p (D_AND MA MB) t ∩
    DFT_event p (D_AND P B) t ∩
    DFT_event p (D_AND (D_AND PA PB) PS_a) t) -
```

```
prob p
  (DFT_event p CS t ∩ DFT_event p SS t ∩
   DFT_event p (D_AND MA (D_BEFORE MS MA)) t ∩
   DFT_event p (D_AND MA MB) t ∩
   DFT_event p (D_AND P B) t ∩
   DFT_event p (D_AND (D_AND PA PB) PS_a) t))
```

[prob_prod_10_of_10]

```
⊢ ∀p M X ii A s1 s2 s3 s4 s5 s6 s7 s8 s9 s10.
     ALL_DISTINCT [s1; s2; s3; s4; s5; s6; s7; s8; s9; s10] ∧
     prob_space p ∧ indep_vars p M X ii ∧
     {s1; s2; s3; s4; s5; s6; s7; s8; s9; s10} ⊆ ii ∧
     (∀i.
          i ∈ {s1; s2; s3; s4; s5; s6; s7; s8; s9; s10} ⇒
          A i ∈ measurable_sets (M i)) ⇒
     (prob p
        (PREIMAGE (X s1) (A s1) ∩ p_space p ∩
         (PREIMAGE (X s2) (A s2) ∩ p_space p) ∩
         (PREIMAGE (X s3) (A s3) ∩ p_space p) ∩
         (PREIMAGE (X s4) (A s4) ∩ p_space p) ∩
         (PREIMAGE (X s5) (A s5) ∩ p_space p) ∩
         (PREIMAGE (X s6) (A s6) ∩ p_space p) ∩
         (PREIMAGE (X s7) (A s7) ∩ p_space p) ∩
         (PREIMAGE (X s8) (A s8) ∩ p_space p) ∩
         (PREIMAGE (X s9) (A s9) ∩ p_space p) ∩
         (PREIMAGE (X s10) (A s10) ∩ p_space p)) =
      prob p (PREIMAGE (X s1) (A s1) ∩ p_space p) ×
      prob p (PREIMAGE (X s2) (A s2) ∩ p_space p) ×
      prob p (PREIMAGE (X s3) (A s3) ∩ p_space p) ×
      prob p (PREIMAGE (X s4) (A s4) ∩ p_space p) ×
      prob p (PREIMAGE (X s5) (A s5) ∩ p_space p) ×
      prob p (PREIMAGE (X s6) (A s6) ∩ p_space p) ×
      prob p (PREIMAGE (X s7) (A s7) ∩ p_space p) ×
      prob p (PREIMAGE (X s8) (A s8) ∩ p_space p) ×
      prob p (PREIMAGE (X s9) (A s9) ∩ p_space p) ×
      prob p (PREIMAGE (X s10) (A s10) ∩ p_space p))
```

[prob_prod_any_2_of_10]

```
⊢ ∀p M X ii A s t.
     s ≠ t ∧ prob_space p ∧ indep_vars p M X ii ∧
     {s; t} ⊆ ii ∧
     (∀i. i ∈ {s; t} ⇒ A i ∈ measurable_sets (M i)) ⇒
     (prob p
        (PREIMAGE (X s) (A s) ∩ p_space p ∩
         (PREIMAGE (X t) (A t) ∩ p_space p)) =
      prob p (PREIMAGE (X s) (A s) ∩ p_space p) ×
      prob p (PREIMAGE (X t) (A t) ∩ p_space p))
```

[prob_prod_any_2_of_10_set]

```
⊢ ∀p ff ii A s t.
      s ≠ t ∧ prob_space p ∧ indep_sets p ff ii ∧ {s; t} ⊆ ii ∧
      (∀i. i ∈ {s; t} ⇒ A i ∈ ff i) ∧
      (∀i. i ∈ {s; t} ⇒ A i ∈ events p) ⇒
      (prob p (A s ∩ A t) = prob p (A s) × prob p (A t))
```

[prob_prod_any_3_of_10]

```
⊢ ∀p M X ii A s t k.
      ALL_DISTINCT [s; t; k] ∧ prob_space p ∧
      indep_vars p M X ii ∧ {s; t; k} ⊆ ii ∧
      (∀i. i ∈ {s; t; k} ⇒ A i ∈ measurable_sets (M i)) ⇒
      (prob p
         (PREIMAGE (X s) (A s) ∩ p_space p ∩
          (PREIMAGE (X t) (A t) ∩ p_space p) ∩
          (PREIMAGE (X k) (A k) ∩ p_space p)) =
       prob p (PREIMAGE (X s) (A s) ∩ p_space p) ×
       prob p (PREIMAGE (X t) (A t) ∩ p_space p) ×
       prob p (PREIMAGE (X k) (A k) ∩ p_space p))
```

[prob_prod_any_3_of_10_set]

```
⊢ ∀p ff ii A s t k.
      ALL_DISTINCT [s; t; k] ∧ prob_space p ∧
      indep_sets p ff ii ∧ {s; t; k} ⊆ ii ∧
      (∀i. i ∈ {s; t; k} ⇒ A i ∈ ff i) ∧
      (∀i. i ∈ {s; t; k} ⇒ A i ∈ events p) ⇒
      (prob p (A s ∩ A t ∩ A k) =
       prob p (A s) × prob p (A t) × prob p (A k))
```

[prob_prod_any_4_of_10]

```
⊢ ∀p M X ii A s1 s2 s3 s4.
      ALL_DISTINCT [s1; s2; s3; s4] ∧ prob_space p ∧
      indep_vars p M X ii ∧ {s1; s2; s3; s4} ⊆ ii ∧
      (∀i. i ∈ {s1; s2; s3; s4} ⇒ A i ∈ measurable_sets (M i)) ⇒
      (prob p
         (PREIMAGE (X s1) (A s1) ∩ p_space p ∩
          (PREIMAGE (X s2) (A s2) ∩ p_space p) ∩
          (PREIMAGE (X s3) (A s3) ∩ p_space p) ∩
          (PREIMAGE (X s4) (A s4) ∩ p_space p)) =
       prob p (PREIMAGE (X s1) (A s1) ∩ p_space p) ×
       prob p (PREIMAGE (X s2) (A s2) ∩ p_space p) ×
       prob p (PREIMAGE (X s3) (A s3) ∩ p_space p) ×
       prob p (PREIMAGE (X s4) (A s4) ∩ p_space p))
```

[prob_prod_any_4_of_10_set]

```
⊢ ∀p ff ii A s1 s2 s3 s4.
      ALL_DISTINCT [s1; s2; s3; s4] ∧ prob_space p ∧
      indep_sets p ff ii ∧ {s1; s2; s3; s4} ⊆ ii ∧
```

$(\forall i.\ i \in \{s_1;\ s_2;\ s_3;\ s_4\} \Rightarrow A\ i \in ff\ i) \wedge$
$(\forall i.\ i \in \{s_1;\ s_2;\ s_3;\ s_4\} \Rightarrow A\ i \in$ events $p) \Rightarrow$
(prob $p$ $(A\ s_1 \cap A\ s_2 \cap A\ s_3 \cap A\ s_4)$ =
 prob $p$ $(A\ s_1)$ × prob $p$ $(A\ s_2)$ × prob $p$ $(A\ s_3)$ ×
 prob $p$ $(A\ s_4)$)

[prob_prod_any_5_of_10]

⊢ $\forall p\ M\ X\ ii\ A\ s_1\ s_2\ s_3\ s_4\ s_5.$
ALL_DISTINCT $[s_1;\ s_2;\ s_3;\ s_4;\ s_5]$ ∧ prob_space $p$ ∧
indep_vars $p\ M\ X\ ii \wedge \{s_1;\ s_2;\ s_3;\ s_4;\ s_5\} \subseteq ii \wedge$
$(\forall i.$
  $i \in \{s_1;\ s_2;\ s_3;\ s_4;\ s_5\} \Rightarrow$
  $A\ i \in$ measurable_sets $(M\ i)) \Rightarrow$
(prob $p$
 (PREIMAGE $(X\ s_1)$ $(A\ s_1)$ ∩ p_space $p$ ∩
 (PREIMAGE $(X\ s_2)$ $(A\ s_2)$ ∩ p_space $p$) ∩
 (PREIMAGE $(X\ s_3)$ $(A\ s_3)$ ∩ p_space $p$) ∩
 (PREIMAGE $(X\ s_4)$ $(A\ s_4)$ ∩ p_space $p$) ∩
 (PREIMAGE $(X\ s_5)$ $(A\ s_5)$ ∩ p_space $p$)) =
 prob $p$ (PREIMAGE $(X\ s_1)$ $(A\ s_1)$ ∩ p_space $p$) ×
 prob $p$ (PREIMAGE $(X\ s_2)$ $(A\ s_2)$ ∩ p_space $p$) ×
 prob $p$ (PREIMAGE $(X\ s_3)$ $(A\ s_3)$ ∩ p_space $p$) ×
 prob $p$ (PREIMAGE $(X\ s_4)$ $(A\ s_4)$ ∩ p_space $p$) ×
 prob $p$ (PREIMAGE $(X\ s_5)$ $(A\ s_5)$ ∩ p_space $p$))

[prob_prod_any_5_of_10_set]

⊢ $\forall p\ ff\ ii\ A\ s_1\ s_2\ s_3\ s_4\ s_5.$
ALL_DISTINCT $[s_1;\ s_2;\ s_3;\ s_4;\ s_5]$ ∧ prob_space $p$ ∧
indep_sets $p\ ff\ ii \wedge \{s_1;\ s_2;\ s_3;\ s_4;\ s_5\} \subseteq ii \wedge$
$(\forall i.\ i \in \{s_1;\ s_2;\ s_3;\ s_4;\ s_5\} \Rightarrow A\ i \in ff\ i) \wedge$
$(\forall i.\ i \in \{s_1;\ s_2;\ s_3;\ s_4;\ s_5\} \Rightarrow A\ i \in$ events $p) \Rightarrow$
(prob $p$ $(A\ s_1 \cap A\ s_2 \cap A\ s_3 \cap A\ s_4 \cap A\ s_5)$ =
 prob $p$ $(A\ s_1)$ × prob $p$ $(A\ s_2)$ × prob $p$ $(A\ s_3)$ ×
 prob $p$ $(A\ s_4)$ × prob $p$ $(A\ s_5)$)

[prob_prod_any_6_of_10]

⊢ $\forall p\ M\ X\ ii\ A\ s_1\ s_2\ s_3\ s_4\ s_5\ s_6.$
ALL_DISTINCT $[s_1;\ s_2;\ s_3;\ s_4;\ s_5;\ s_6]$ ∧ prob_space $p$ ∧
indep_vars $p\ M\ X\ ii \wedge \{s_1;\ s_2;\ s_3;\ s_4;\ s_5;\ s_6\} \subseteq ii \wedge$
$(\forall i.$
  $i \in \{s_1;\ s_2;\ s_3;\ s_4;\ s_5;\ s_6\} \Rightarrow$
  $A\ i \in$ measurable_sets $(M\ i)) \Rightarrow$
(prob $p$
 (PREIMAGE $(X\ s_1)$ $(A\ s_1)$ ∩ p_space $p$ ∩
 (PREIMAGE $(X\ s_2)$ $(A\ s_2)$ ∩ p_space $p$) ∩
 (PREIMAGE $(X\ s_3)$ $(A\ s_3)$ ∩ p_space $p$) ∩
 (PREIMAGE $(X\ s_4)$ $(A\ s_4)$ ∩ p_space $p$) ∩
 (PREIMAGE $(X\ s_5)$ $(A\ s_5)$ ∩ p_space $p$) ∩

```
    (PREIMAGE (X s6) (A s6) ∩ p_space p)) =
   prob p (PREIMAGE (X s1) (A s1) ∩ p_space p) ×
   prob p (PREIMAGE (X s2) (A s2) ∩ p_space p) ×
   prob p (PREIMAGE (X s3) (A s3) ∩ p_space p) ×
   prob p (PREIMAGE (X s4) (A s4) ∩ p_space p) ×
   prob p (PREIMAGE (X s5) (A s5) ∩ p_space p) ×
   prob p (PREIMAGE (X s6) (A s6) ∩ p_space p))
```

[prob_prod_any_6_of_10_set]

```
⊢ ∀p ff ii A s1 s2 s3 s4 s5 s6.
    ALL_DISTINCT [s1; s2; s3; s4; s5; s6] ∧ prob_space p ∧
    indep_sets p ff ii ∧ {s1; s2; s3; s4; s5; s6} ⊆ ii ∧
    (∀i. i ∈ {s1; s2; s3; s4; s5; s6} ⇒ A i ∈ ff i) ∧
    (∀i. i ∈ {s1; s2; s3; s4; s5; s6} ⇒ A i ∈ events p) ⇒
    (prob p (A s1 ∩ A s2 ∩ A s3 ∩ A s4 ∩ A s5 ∩ A s6) =
     prob p (A s1) × prob p (A s2) × prob p (A s3) ×
     prob p (A s4) × prob p (A s5) × prob p (A s6))
```

[prob_prod_any_7_of_10]

```
⊢ ∀p M X ii A s1 s2 s3 s4 s5 s6 s7.
    ALL_DISTINCT [s1; s2; s3; s4; s5; s6; s7] ∧
    prob_space p ∧ indep_vars p M X ii ∧
    {s1; s2; s3; s4; s5; s6; s7} ⊆ ii ∧
    (∀i.
       i ∈ {s1; s2; s3; s4; s5; s6; s7} ⇒
       A i ∈ measurable_sets (M i)) ⇒
    (prob p
      (PREIMAGE (X s1) (A s1) ∩ p_space p ∩
       (PREIMAGE (X s2) (A s2) ∩ p_space p) ∩
       (PREIMAGE (X s3) (A s3) ∩ p_space p) ∩
       (PREIMAGE (X s4) (A s4) ∩ p_space p) ∩
       (PREIMAGE (X s5) (A s5) ∩ p_space p) ∩
       (PREIMAGE (X s6) (A s6) ∩ p_space p) ∩
       (PREIMAGE (X s7) (A s7) ∩ p_space p)) =
     prob p (PREIMAGE (X s1) (A s1) ∩ p_space p) ×
     prob p (PREIMAGE (X s2) (A s2) ∩ p_space p) ×
     prob p (PREIMAGE (X s3) (A s3) ∩ p_space p) ×
     prob p (PREIMAGE (X s4) (A s4) ∩ p_space p) ×
     prob p (PREIMAGE (X s5) (A s5) ∩ p_space p) ×
     prob p (PREIMAGE (X s6) (A s6) ∩ p_space p) ×
     prob p (PREIMAGE (X s7) (A s7) ∩ p_space p))
```

[prob_prod_any_7_of_10_set]

```
⊢ ∀p ff ii A s1 s2 s3 s4 s5 s6 s7.
    ALL_DISTINCT [s1; s2; s3; s4; s5; s6; s7] ∧
    prob_space p ∧ indep_sets p ff ii ∧
    {s1; s2; s3; s4; s5; s6; s7} ⊆ ii ∧
    (∀i. i ∈ {s1; s2; s3; s4; s5; s6; s7} ⇒ A i ∈ ff i) ∧
```

$(\forall i.\ i \in \{s_1;\ s_2;\ s_3;\ s_4;\ s_5;\ s_6;\ s_7\} \Rightarrow A\ i \in \texttt{events}\ p) \Rightarrow$
$(\texttt{prob}\ p\ (A\ s_1 \cap A\ s_2 \cap A\ s_3 \cap A\ s_4 \cap A\ s_5 \cap A\ s_6 \cap A\ s_7) =$
$\texttt{prob}\ p\ (A\ s_1) \times \texttt{prob}\ p\ (A\ s_2) \times \texttt{prob}\ p\ (A\ s_3) \times$
$\texttt{prob}\ p\ (A\ s_4) \times \texttt{prob}\ p\ (A\ s_5) \times \texttt{prob}\ p\ (A\ s_6) \times$
$\texttt{prob}\ p\ (A\ s_7))$

[prob_prod_any_8_of_10]

$\vdash \forall p\ M\ X\ ii\ A\ s_1\ s_2\ s_3\ s_4\ s_5\ s_6\ s_7\ s_8.$
$\texttt{ALL\_DISTINCT}\ [s_1;\ s_2;\ s_3;\ s_4;\ s_5;\ s_6;\ s_7;\ s_8] \wedge$
$\texttt{prob\_space}\ p \wedge \texttt{indep\_vars}\ p\ M\ X\ ii \wedge$
$\{s_1;\ s_2;\ s_3;\ s_4;\ s_5;\ s_6;\ s_7;\ s_8\} \subseteq ii \wedge$
$(\forall i.$
$i \in \{s_1;\ s_2;\ s_3;\ s_4;\ s_5;\ s_6;\ s_7;\ s_8\} \Rightarrow$
$A\ i \in \texttt{measurable\_sets}\ (M\ i)) \Rightarrow$
$(\texttt{prob}\ p$
$(\texttt{PREIMAGE}\ (X\ s_1)\ (A\ s_1) \cap \texttt{p\_space}\ p \cap$
$(\texttt{PREIMAGE}\ (X\ s_2)\ (A\ s_2) \cap \texttt{p\_space}\ p) \cap$
$(\texttt{PREIMAGE}\ (X\ s_3)\ (A\ s_3) \cap \texttt{p\_space}\ p) \cap$
$(\texttt{PREIMAGE}\ (X\ s_4)\ (A\ s_4) \cap \texttt{p\_space}\ p) \cap$
$(\texttt{PREIMAGE}\ (X\ s_5)\ (A\ s_5) \cap \texttt{p\_space}\ p) \cap$
$(\texttt{PREIMAGE}\ (X\ s_6)\ (A\ s_6) \cap \texttt{p\_space}\ p) \cap$
$(\texttt{PREIMAGE}\ (X\ s_7)\ (A\ s_7) \cap \texttt{p\_space}\ p) \cap$
$(\texttt{PREIMAGE}\ (X\ s_8)\ (A\ s_8) \cap \texttt{p\_space}\ p)) =$
$\texttt{prob}\ p\ (\texttt{PREIMAGE}\ (X\ s_1)\ (A\ s_1) \cap \texttt{p\_space}\ p) \times$
$\texttt{prob}\ p\ (\texttt{PREIMAGE}\ (X\ s_2)\ (A\ s_2) \cap \texttt{p\_space}\ p) \times$
$\texttt{prob}\ p\ (\texttt{PREIMAGE}\ (X\ s_3)\ (A\ s_3) \cap \texttt{p\_space}\ p) \times$
$\texttt{prob}\ p\ (\texttt{PREIMAGE}\ (X\ s_4)\ (A\ s_4) \cap \texttt{p\_space}\ p) \times$
$\texttt{prob}\ p\ (\texttt{PREIMAGE}\ (X\ s_5)\ (A\ s_5) \cap \texttt{p\_space}\ p) \times$
$\texttt{prob}\ p\ (\texttt{PREIMAGE}\ (X\ s_6)\ (A\ s_6) \cap \texttt{p\_space}\ p) \times$
$\texttt{prob}\ p\ (\texttt{PREIMAGE}\ (X\ s_7)\ (A\ s_7) \cap \texttt{p\_space}\ p) \times$
$\texttt{prob}\ p\ (\texttt{PREIMAGE}\ (X\ s_8)\ (A\ s_8) \cap \texttt{p\_space}\ p))$

[prob_prod_any_8_of_10_set]

$\vdash \forall p\ ff\ ii\ A\ s_1\ s_2\ s_3\ s_4\ s_5\ s_6\ s_7\ s_8.$
$\texttt{ALL\_DISTINCT}\ [s_1;\ s_2;\ s_3;\ s_4;\ s_5;\ s_6;\ s_7;\ s_8] \wedge$
$\texttt{prob\_space}\ p \wedge \texttt{indep\_sets}\ p\ ff\ ii \wedge$
$\{s_1;\ s_2;\ s_3;\ s_4;\ s_5;\ s_6;\ s_7;\ s_8\} \subseteq ii \wedge$
$(\forall i.\ i \in \{s_1;\ s_2;\ s_3;\ s_4;\ s_5;\ s_6;\ s_7;\ s_8\} \Rightarrow A\ i \in ff\ i) \wedge$
$(\forall i.$
$i \in \{s_1;\ s_2;\ s_3;\ s_4;\ s_5;\ s_6;\ s_7;\ s_8\} \Rightarrow$
$A\ i \in \texttt{events}\ p) \Rightarrow$
$(\texttt{prob}\ p$
$(A\ s_1 \cap A\ s_2 \cap A\ s_3 \cap A\ s_4 \cap A\ s_5 \cap A\ s_6 \cap A\ s_7 \cap A\ s_8) =$
$\texttt{prob}\ p\ (A\ s_1) \times \texttt{prob}\ p\ (A\ s_2) \times \texttt{prob}\ p\ (A\ s_3) \times$
$\texttt{prob}\ p\ (A\ s_4) \times \texttt{prob}\ p\ (A\ s_5) \times \texttt{prob}\ p\ (A\ s_6) \times$
$\texttt{prob}\ p\ (A\ s_7) \times \texttt{prob}\ p\ (A\ s_8))$

[prob_prod_any_9_of_10]

$\vdash \forall p\ M\ X\ ii\ A\ s_1\ s_2\ s_3\ s_4\ s_5\ s_6\ s_7\ s_8\ s_9.$
ALL_DISTINCT $[s_1;\ s_2;\ s_3;\ s_4;\ s_5;\ s_6;\ s_7;\ s_8;\ s_9] \land$
prob_space $p \land$ indep_vars $p\ M\ X\ ii \land$
$\{s_1;\ s_2;\ s_3;\ s_4;\ s_5;\ s_6;\ s_7;\ s_8;\ s_9\} \subseteq ii \land$
$(\forall i.$
$i \in \{s_1;\ s_2;\ s_3;\ s_4;\ s_5;\ s_6;\ s_7;\ s_8;\ s_9\} \Rightarrow$
$A\ i \in$ measurable_sets $(M\ i)) \Rightarrow$
(prob $p$
(PREIMAGE $(X\ s_1)\ (A\ s_1) \cap$ p_space $p \cap$
(PREIMAGE $(X\ s_2)\ (A\ s_2) \cap$ p_space $p) \cap$
(PREIMAGE $(X\ s_3)\ (A\ s_3) \cap$ p_space $p) \cap$
(PREIMAGE $(X\ s_4)\ (A\ s_4) \cap$ p_space $p) \cap$
(PREIMAGE $(X\ s_5)\ (A\ s_5) \cap$ p_space $p) \cap$
(PREIMAGE $(X\ s_6)\ (A\ s_6) \cap$ p_space $p) \cap$
(PREIMAGE $(X\ s_7)\ (A\ s_7) \cap$ p_space $p) \cap$
(PREIMAGE $(X\ s_8)\ (A\ s_8) \cap$ p_space $p) \cap$
(PREIMAGE $(X\ s_9)\ (A\ s_9) \cap$ p_space $p)) =$
prob $p$ (PREIMAGE $(X\ s_1)\ (A\ s_1) \cap$ p_space $p) \times$
prob $p$ (PREIMAGE $(X\ s_2)\ (A\ s_2) \cap$ p_space $p) \times$
prob $p$ (PREIMAGE $(X\ s_3)\ (A\ s_3) \cap$ p_space $p) \times$
prob $p$ (PREIMAGE $(X\ s_4)\ (A\ s_4) \cap$ p_space $p) \times$
prob $p$ (PREIMAGE $(X\ s_5)\ (A\ s_5) \cap$ p_space $p) \times$
prob $p$ (PREIMAGE $(X\ s_6)\ (A\ s_6) \cap$ p_space $p) \times$
prob $p$ (PREIMAGE $(X\ s_7)\ (A\ s_7) \cap$ p_space $p) \times$
prob $p$ (PREIMAGE $(X\ s_8)\ (A\ s_8) \cap$ p_space $p) \times$
prob $p$ (PREIMAGE $(X\ s_9)\ (A\ s_9) \cap$ p_space $p))

[prob_prod_any_9_of_10_set]

$\vdash \forall p\ ff\ ii\ A\ s_1\ s_2\ s_3\ s_4\ s_5\ s_6\ s_7\ s_8\ s_9.$
ALL_DISTINCT $[s_1;\ s_2;\ s_3;\ s_4;\ s_5;\ s_6;\ s_7;\ s_8;\ s_9] \land$
prob_space $p \land$ indep_sets $p\ ff\ ii \land$
$\{s_1;\ s_2;\ s_3;\ s_4;\ s_5;\ s_6;\ s_7;\ s_8;\ s_9\} \subseteq ii \land$
$(\forall i.$
$i \in \{s_1;\ s_2;\ s_3;\ s_4;\ s_5;\ s_6;\ s_7;\ s_8;\ s_9\} \Rightarrow$
$A\ i \in ff\ i) \land$
$(\forall i.$
$i \in \{s_1;\ s_2;\ s_3;\ s_4;\ s_5;\ s_6;\ s_7;\ s_8;\ s_9\} \Rightarrow$
$A\ i \in$ events $p) \Rightarrow$
(prob $p$
$(A\ s_1 \cap A\ s_2 \cap A\ s_3 \cap A\ s_4 \cap A\ s_5 \cap A\ s_6 \cap A\ s_7 \cap$
$A\ s_8 \cap A\ s_9) =$
prob $p\ (A\ s_1) \times$ prob $p\ (A\ s_2) \times$ prob $p\ (A\ s_3) \times$
prob $p\ (A\ s_4) \times$ prob $p\ (A\ s_5) \times$ prob $p\ (A\ s_6) \times$
prob $p\ (A\ s_7) \times$ prob $p\ (A\ s_8) \times$ prob $p\ (A\ s_9))$

[PRODUCT_UNION_10]

$\vdash \forall s_1\ s_2\ s_3\ s_4\ s_5\ s_6\ s_7\ s_8\ s_9\ s_{10}\ f.$
ALL_DISTINCT $[s_1;\ s_2;\ s_3;\ s_4;\ s_5;\ s_6;\ s_7;\ s_8;\ s_9;\ s_{10}] \Rightarrow$
(product $\{s_1;\ s_2;\ s_3;\ s_4;\ s_5;\ s_6;\ s_7;\ s_8;\ s_9;\ s_{10}\}\ f =$

$f\ s_1 \times f\ s_2 \times f\ s_3 \times f\ s_4 \times f\ s_5 \times f\ s_6 \times f\ s_7 \times f\ s_8 \times$
$f\ s_9 \times f\ s_{10}$)

[PRODUCT_UNION_5]

$\vdash \forall s_1\ s_2\ s_3\ s_4\ s_5\ f$.
ALL_DISTINCT [$s_1$; $s_2$; $s_3$; $s_4$; $s_5$] $\Rightarrow$
(product {$s_1$; $s_2$; $s_3$; $s_4$; $s_5$} $f$ =
$f\ s_1 \times f\ s_2 \times f\ s_3 \times f\ s_4 \times f\ s_5$)

[PRODUCT_UNION_6]

$\vdash \forall s_1\ s_2\ s_3\ s_4\ s_5\ s_6\ f$.
ALL_DISTINCT [$s_1$; $s_2$; $s_3$; $s_4$; $s_5$; $s_6$] $\Rightarrow$
(product {$s_1$; $s_2$; $s_3$; $s_4$; $s_5$; $s_6$} $f$ =
$f\ s_1 \times f\ s_2 \times f\ s_3 \times f\ s_4 \times f\ s_5 \times f\ s_6$)

[PRODUCT_UNION_7]

$\vdash \forall s_1\ s_2\ s_3\ s_4\ s_5\ s_6\ s_7\ f$.
ALL_DISTINCT [$s_1$; $s_2$; $s_3$; $s_4$; $s_5$; $s_6$; $s_7$] $\Rightarrow$
(product {$s_1$; $s_2$; $s_3$; $s_4$; $s_5$; $s_6$; $s_7$} $f$ =
$f\ s_1 \times f\ s_2 \times f\ s_3 \times f\ s_4 \times f\ s_5 \times f\ s_6 \times f\ s_7$)

[PRODUCT_UNION_8]

$\vdash \forall s_1\ s_2\ s_3\ s_4\ s_5\ s_6\ s_7\ s_8\ f$.
ALL_DISTINCT [$s_1$; $s_2$; $s_3$; $s_4$; $s_5$; $s_6$; $s_7$; $s_8$] $\Rightarrow$
(product {$s_1$; $s_2$; $s_3$; $s_4$; $s_5$; $s_6$; $s_7$; $s_8$} $f$ =
$f\ s_1 \times f\ s_2 \times f\ s_3 \times f\ s_4 \times f\ s_5 \times f\ s_6 \times f\ s_7 \times f\ s_8$)

[PRODUCT_UNION_9]

$\vdash \forall s_1\ s_2\ s_3\ s_4\ s_5\ s_6\ s_7\ s_8\ s_9\ f$.
ALL_DISTINCT [$s_1$; $s_2$; $s_3$; $s_4$; $s_5$; $s_6$; $s_7$; $s_8$; $s_9$] $\Rightarrow$
(product {$s_1$; $s_2$; $s_3$; $s_4$; $s_5$; $s_6$; $s_7$; $s_8$; $s_9$} $f$ =
$f\ s_1 \times f\ s_2 \times f\ s_3 \times f\ s_4 \times f\ s_5 \times f\ s_6 \times f\ s_7 \times f\ s_8 \times$
$f\ s_9$)

[sub_lneg]

$\vdash \forall x\ y$.
$x \neq$ NegInf $\wedge$ $y \neq$ NegInf $\vee$ $x \neq$ PosInf $\wedge$ $y \neq$ PosInf $\Rightarrow$
(-$x$ - $y$ = -($x$ + $y$))