Formal Analysis of Electromagnetic Optics

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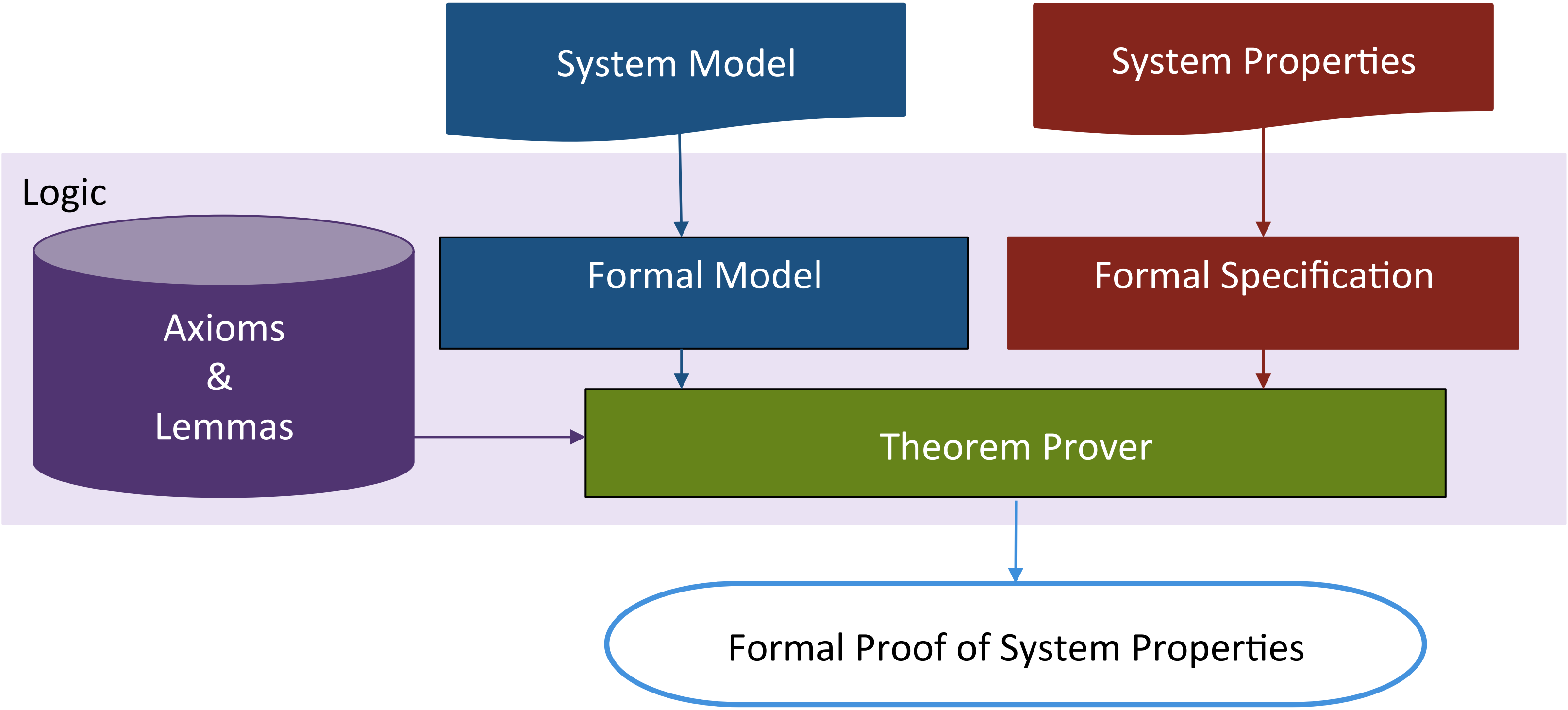
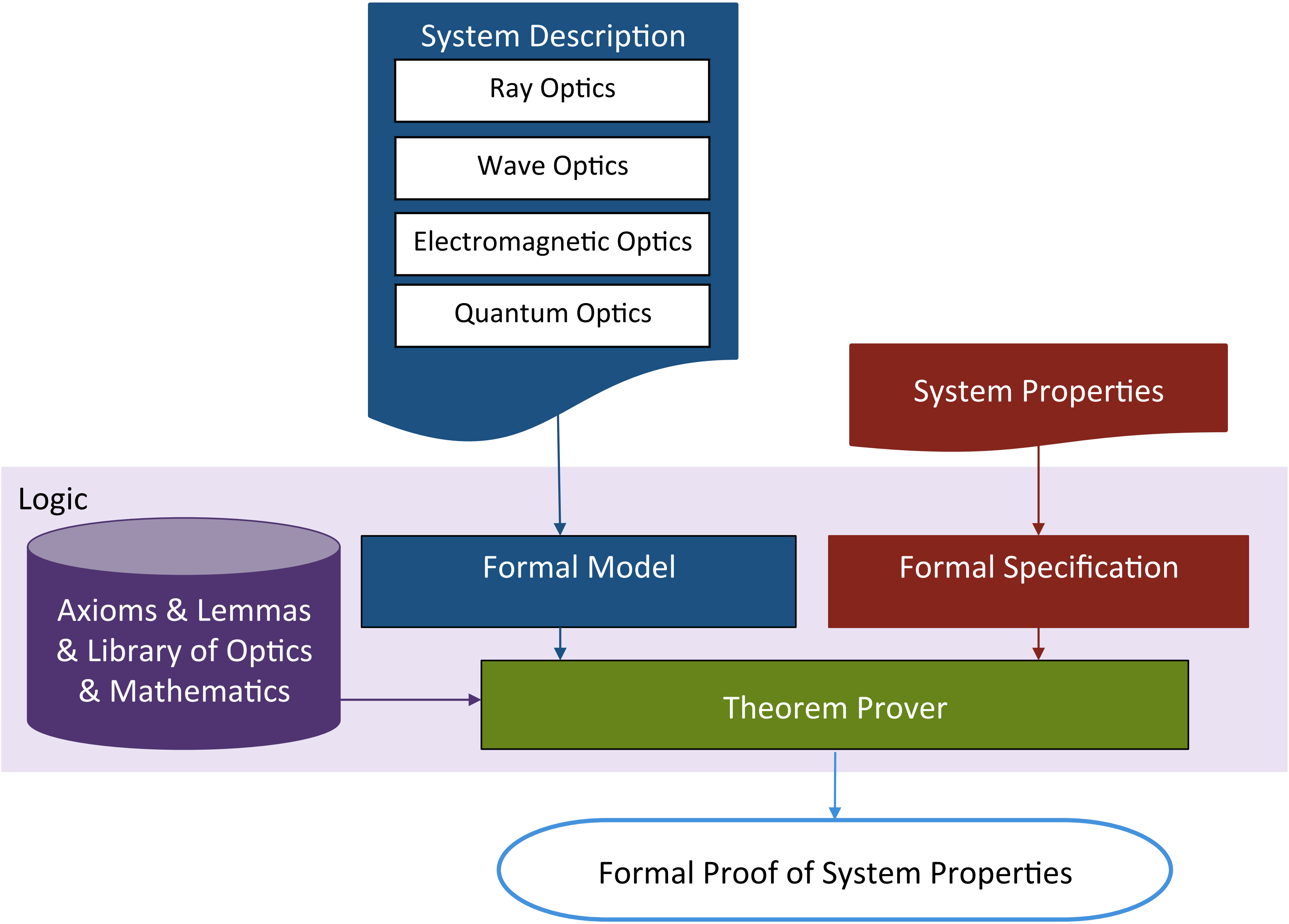
Outline

- Formal Verification
- Methodology
- Formalization of Fabry-Perot Resonators
- Case study: Fabry Perot Resonator and Gas Laser
- Conclusions and Future Works

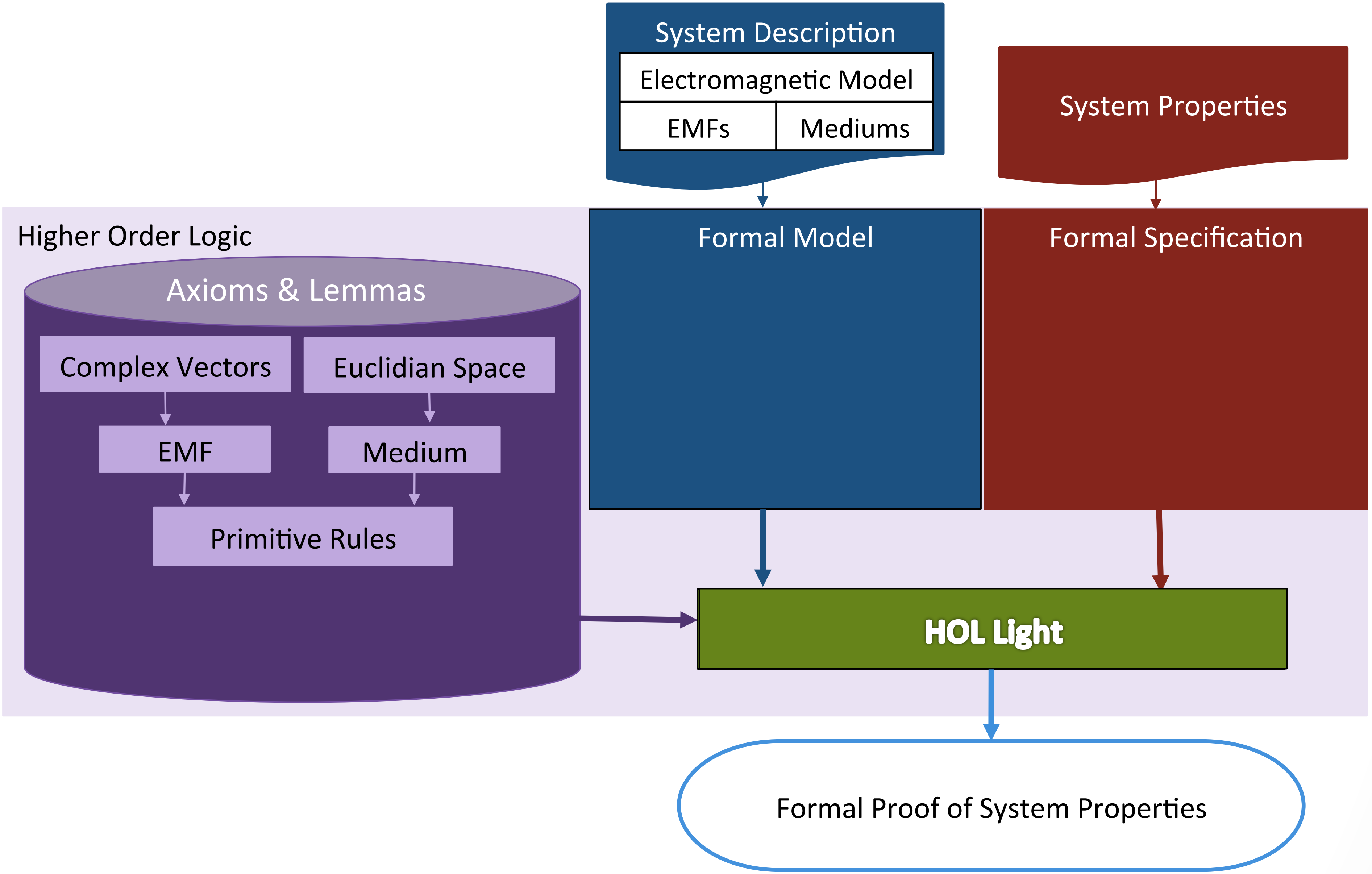
Towards Formal Verification of Optical Systems



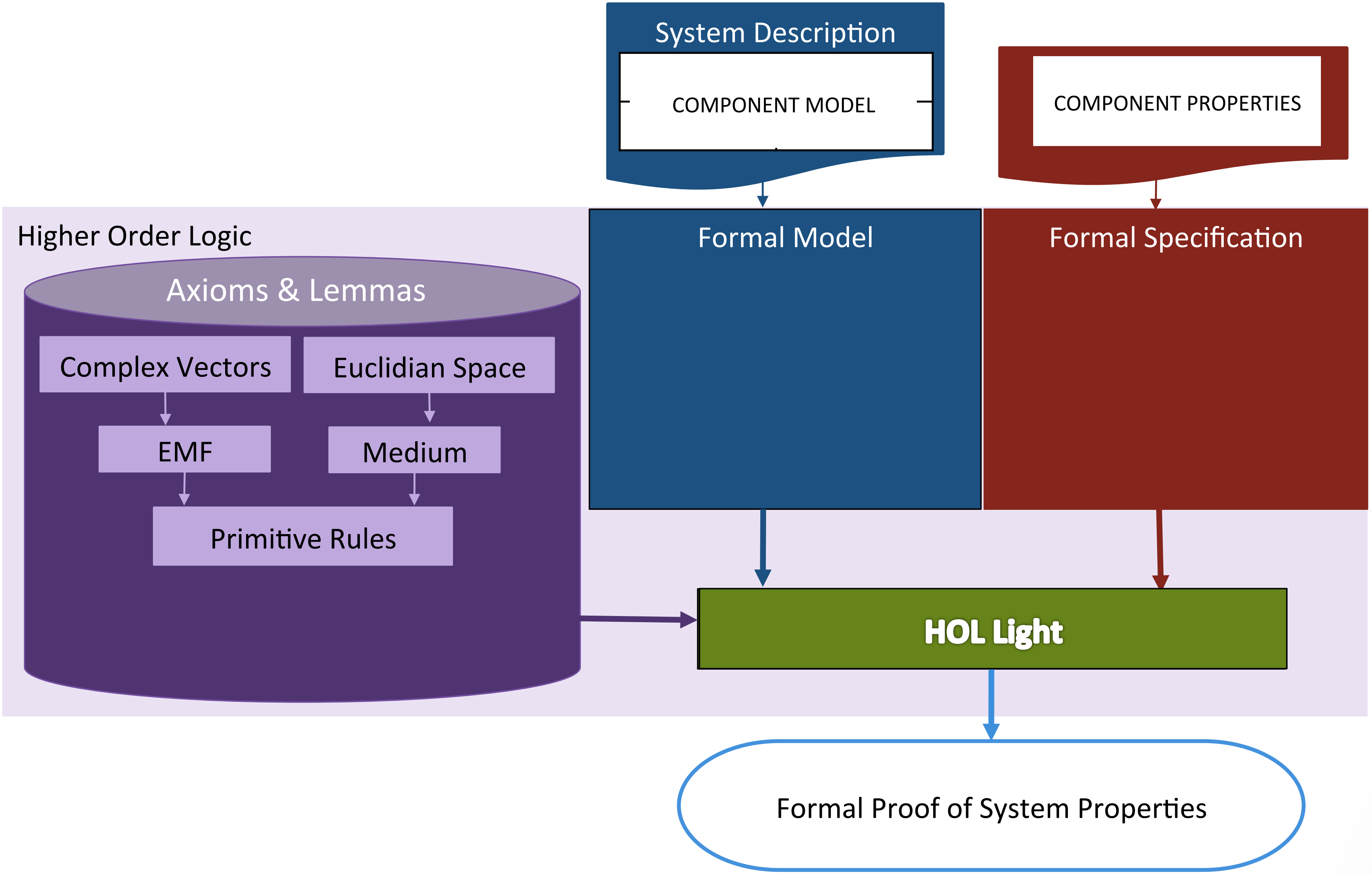
Towards Formal Verification of Optical Systems



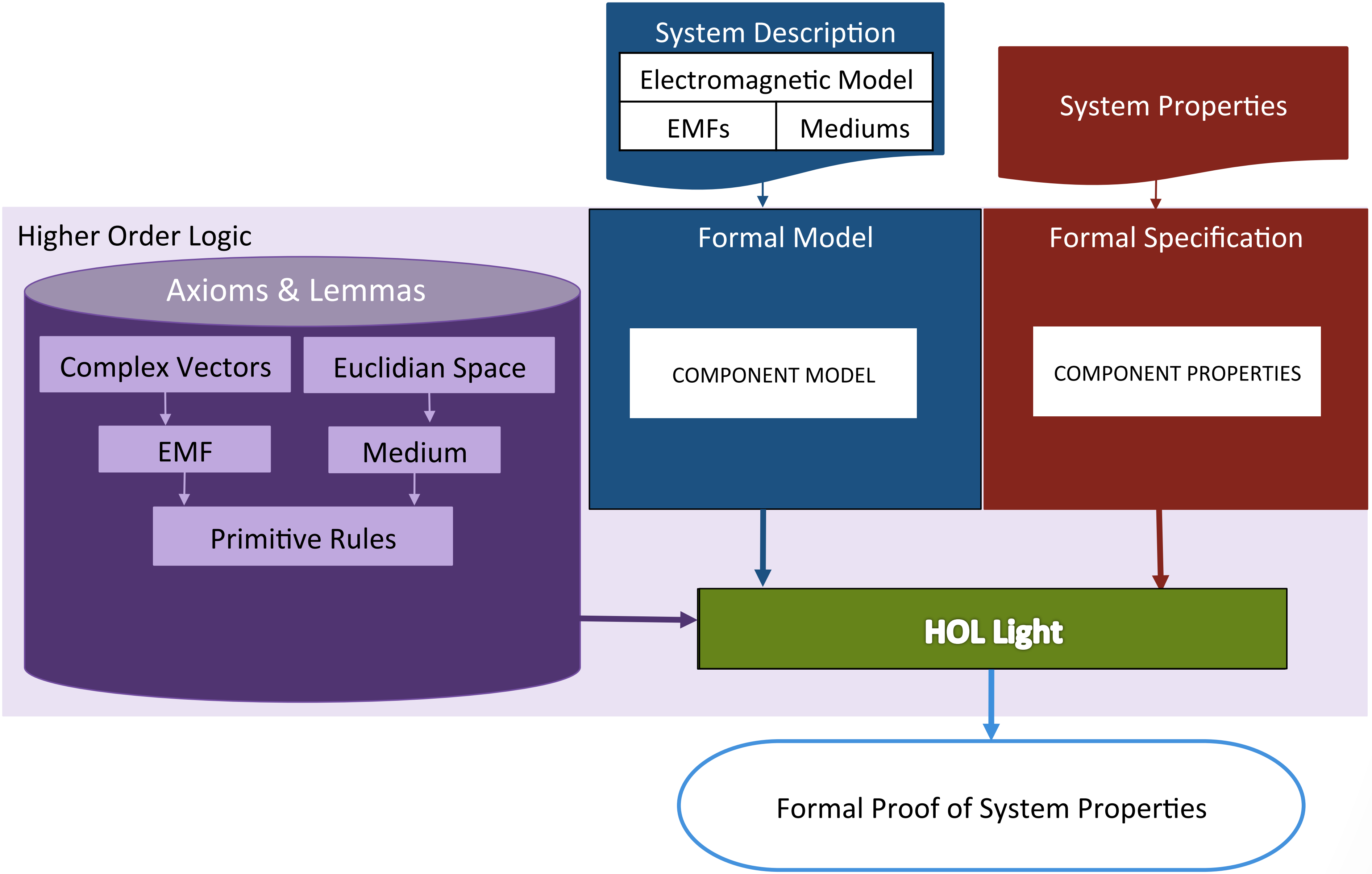
Towards Formalization of Optical Systems



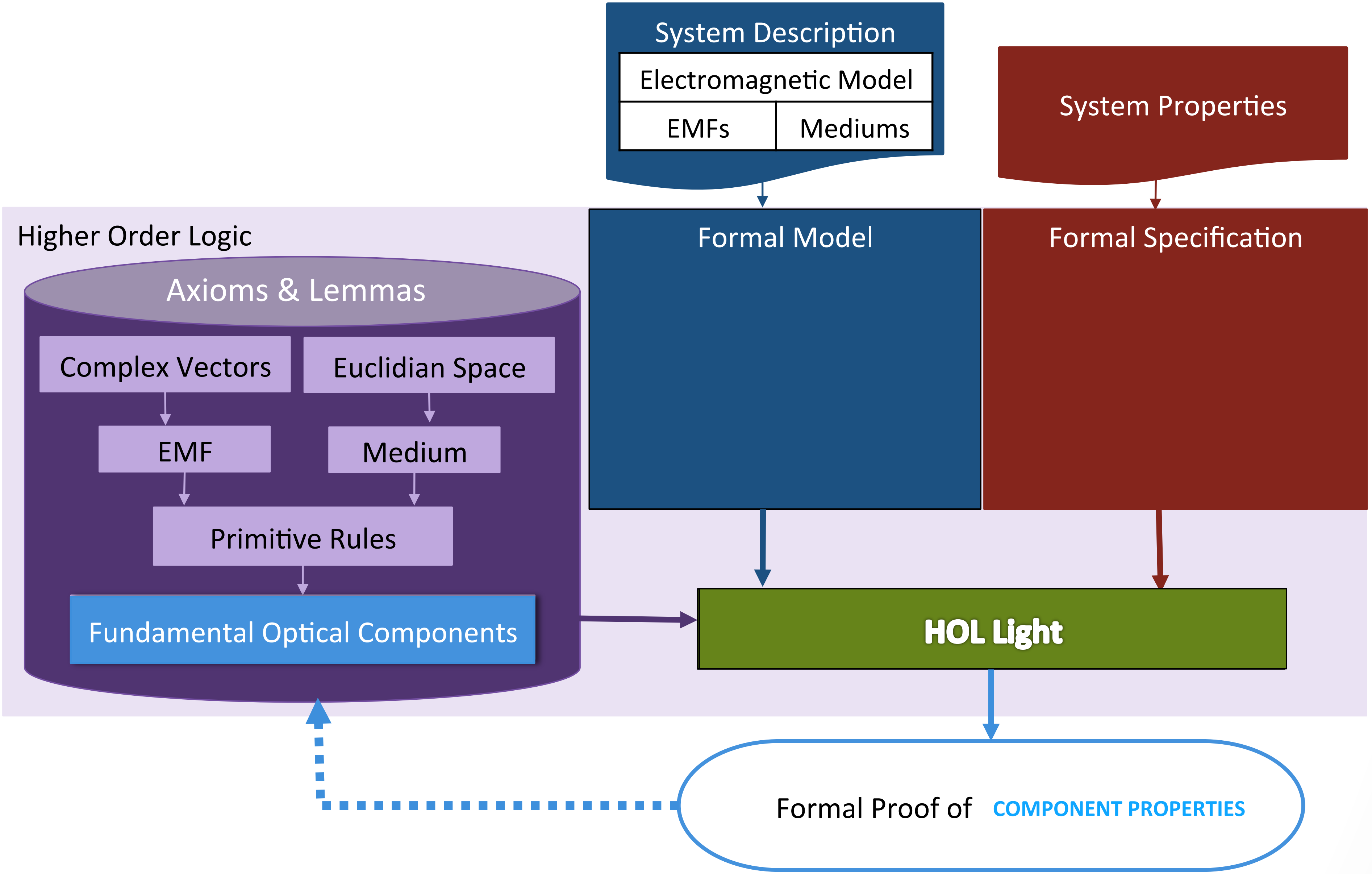
Towards Formalization of Optical Systems



Towards Formalization of Optical Systems



Towards Formalization of Optical Systems



Formalization: Light as Electromagnetic Field

A Field: a physical quantity associated with each point of space-time

Type definition:

$$\texttt{type emf} = \texttt{point} \rightarrow \texttt{time} \rightarrow \texttt{complex}^3 \times \texttt{complex}^3$$

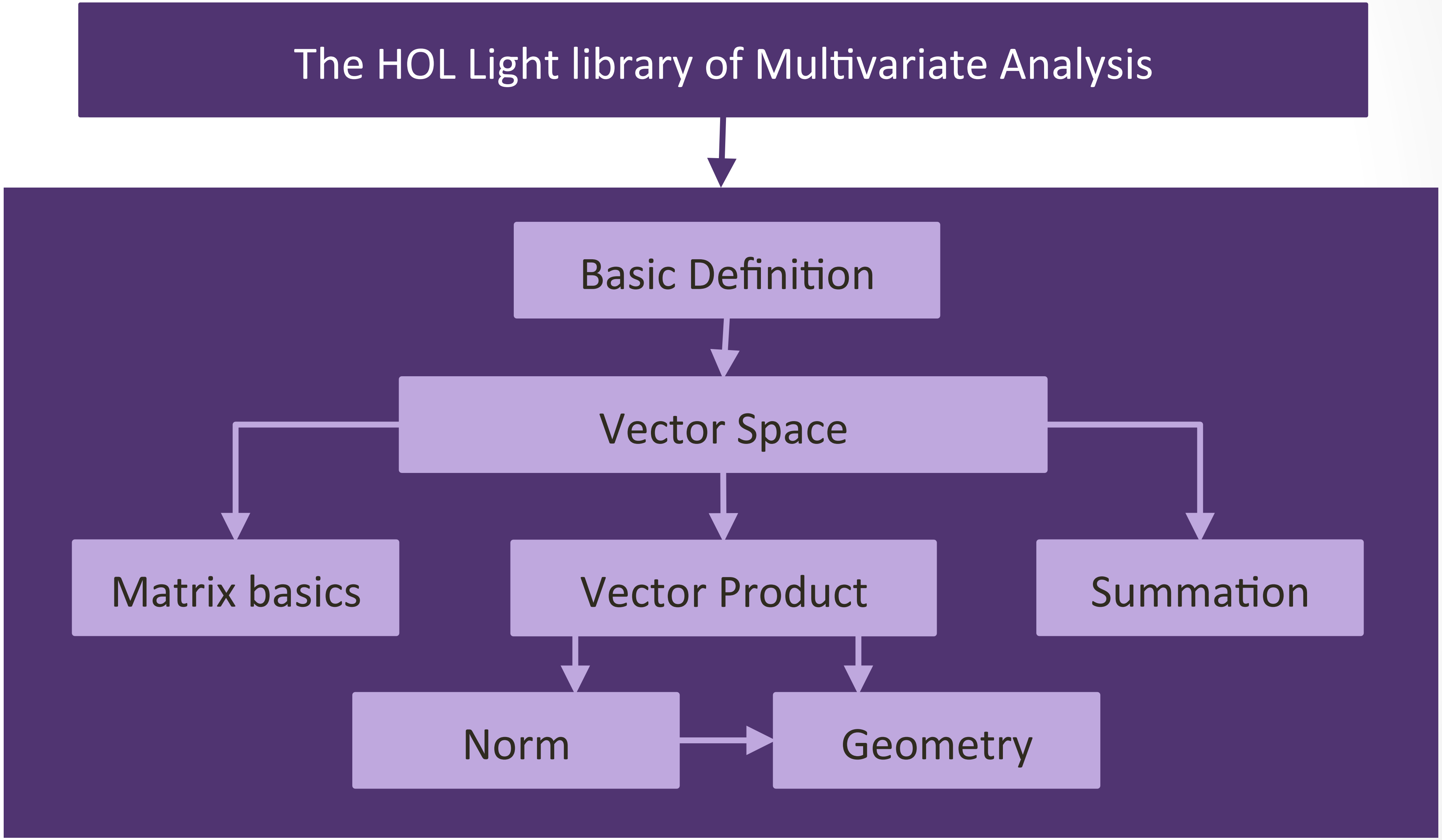
Physical constraints:

- Electric field and Magnetic Fields are Orthogonal

$$\vdash \forall \texttt{emf}.\ \ \texttt{is\_valid\_emf emf} \Leftrightarrow$$
$$(\forall \texttt{r t}.\ \ \texttt{corthogonal}\ (\texttt{e\_of\_emf emf r t})\ (\texttt{h\_of\_emf emf r t}))$$

We need to formally define the concept of orthogonality of complex vectors!

Formalization of Complex Vector Calculus



Formalization: Monochromatic Plane Waves

All the components of the electromagnetic field are harmonic functions of time with the same frequency.

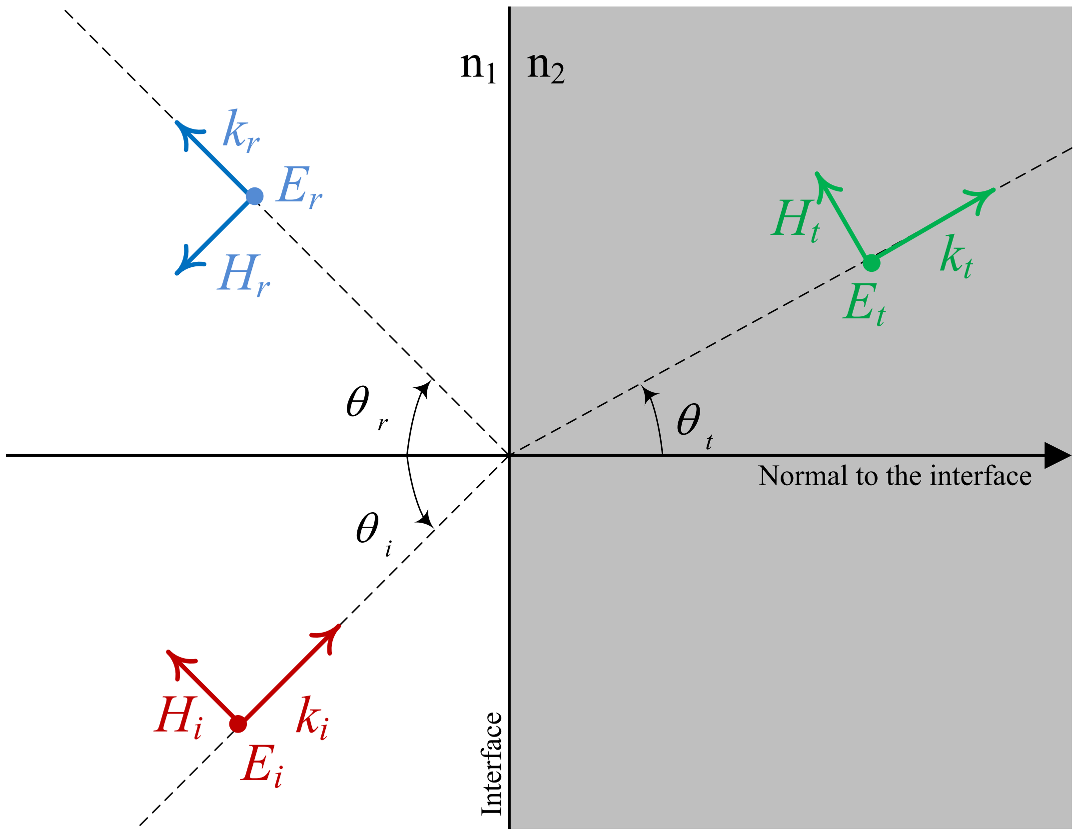
$$\vec{U}(\vec{r}, t) = \vec{a}(\vec{r}) e^{j\phi(\vec{r})} e^{j\omega t}$$

$$\vdash \texttt{plane\_wave}\ (\texttt{k} : \texttt{real}^3)\ (\omega : \texttt{real})\ (\texttt{E} : \texttt{complex}^3)\ (\texttt{H} : \texttt{complex}^3) : \texttt{emf}$$
$$= \lambda(\texttt{r} : \texttt{point})\ (\texttt{t} : \texttt{time}).\ (\texttt{e}^{-\texttt{ii}(\texttt{k}\cdot\texttt{r}-\omega\texttt{t})}\texttt{E}, \texttt{e}^{-\texttt{ii}(\texttt{k}\cdot\texttt{r}-\omega\texttt{t})}\texttt{H})$$

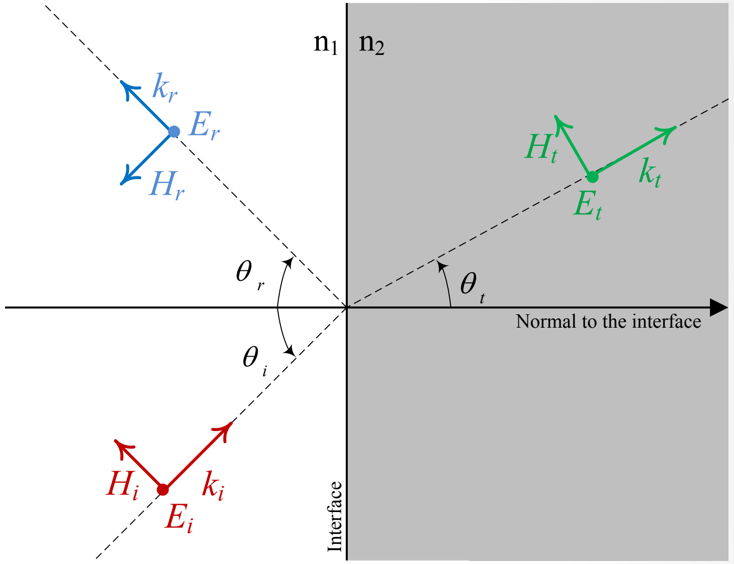
Physical constraints:

$$\vdash \forall \texttt{emf}.\ \texttt{is\_valid\_wave wave} \Leftrightarrow$$
$$(\texttt{is\_valid\_emf wave} \wedge$$
$$(\exists \texttt{k w e h}.$$
$$\&0 < \texttt{w}\ \wedge\ \neg(\texttt{k} = \texttt{vec}\ 0)\ \wedge\ \texttt{wave} = \texttt{plane\_wave k w e h} \wedge$$
$$\texttt{corthogonal e k}\ \wedge\ \texttt{corthogonal h k})$$

Formalization: Plane Interface



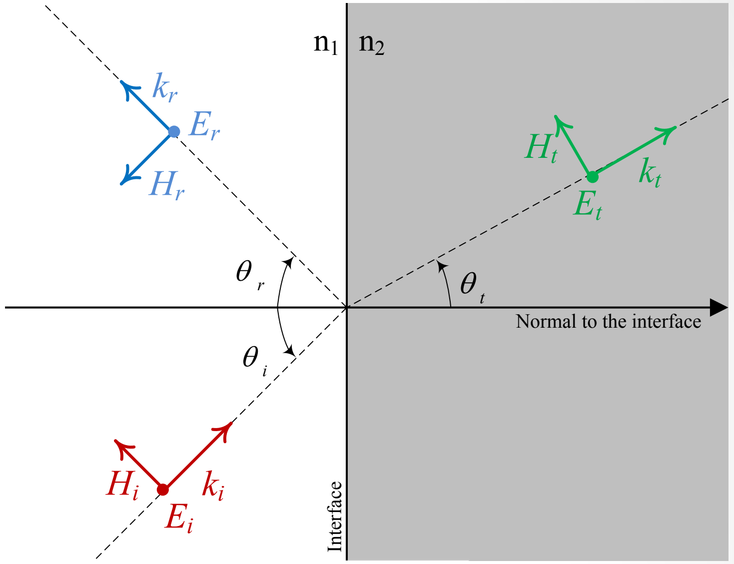
Formalization: Plane Interface



Physical Constraint: Plane wave at Plane Interface

$$
\begin{aligned}
\vdash_{def}\ & \mathtt{is\_plane\_wave\_at\_int\ i\ emf_i\ emf_r\ emf_t} \Leftrightarrow \\
& \mathtt{is\_valid\_interface\ i} \ \wedge\ \mathtt{non\_null\ emf_i} \ \wedge \\
& \mathtt{is\_plane\_wave\ emf_i} \ \wedge\ \mathtt{is\_plane\_wave\ emf_r} \ \wedge\ \mathtt{is\_plane\_wave\ emf_t} \ \wedge \\
& \mathtt{let}\ (n_1, n_2, p, n) = \mathtt{i\ in} \\
& \mathtt{let}\ (k_i, k_r, k_t) = \mathtt{map\_trpl\ k\_of\_w}\ (\mathtt{emf_i}, \mathtt{emf_r}, \mathtt{emf_t})\ \mathtt{in} \\
& \mathtt{let}\ (e_i, e_r, e_t) = \mathtt{map\_trpl}\ (\mathtt{norm} \circ \mathtt{e\_of\_w})\ (\mathtt{emf_i}, \mathtt{emf_r}, \mathtt{emf_t})\ \mathtt{in} \\
& \mathtt{let}\ (h_i, h_r, h_t) = \mathtt{map\_trpl}\ (\mathtt{norm} \circ \mathtt{k\_of\_w})\ (\mathtt{emf_i}, \mathtt{emf_r}, \mathtt{emf_t})\ \mathtt{in} \\
& 0 \le (k_i \cdot n) \ \wedge\ (k_r \cdot n) \le 0 \ \wedge\ 0 \le (k_t \cdot n) \ \wedge \\
& (\forall pt.\ \ pt \in p \Rightarrow \forall t.\ \ \mathtt{boundary\_conditions}\ (\mathtt{emf_i} + \mathtt{emf_r})\ \mathtt{emf_t}\ n\ pt\ t) \ \wedge \\
& \exists k_0.\ \ \mathtt{norm}\ k_i = k_0 n_1 \ \wedge\ \mathtt{norm}\ k_r = k_0 n_1 \ \wedge\ \mathtt{norm}\ k_t = k_0 n_2 \ \wedge \\
& \exists \eta_0.\ \ h_i = e_i n_1/\eta_0 \ \wedge\ h_r = e_r n_1/\eta_0 \ \wedge\ h_t = e_t n_2/\eta_0
\end{aligned}
$$

Formalization: Plane Interface



Physical Constraint: Plane wave at Plane Interface

$\vdash_{def}$ is_plane_wave_at_int i emf_i emf_r emf_t $\Leftrightarrow$
is_valid_interface i $\wedge$ non_null emf_i $\wedge$
is_plane_wave emf_i $\wedge$ is_plane_wave emf_r $\wedge$ is_plane_wave emf_t $\wedge$

type interface = medium # medium # plane # real^3

"medium" indicates the refractive index.

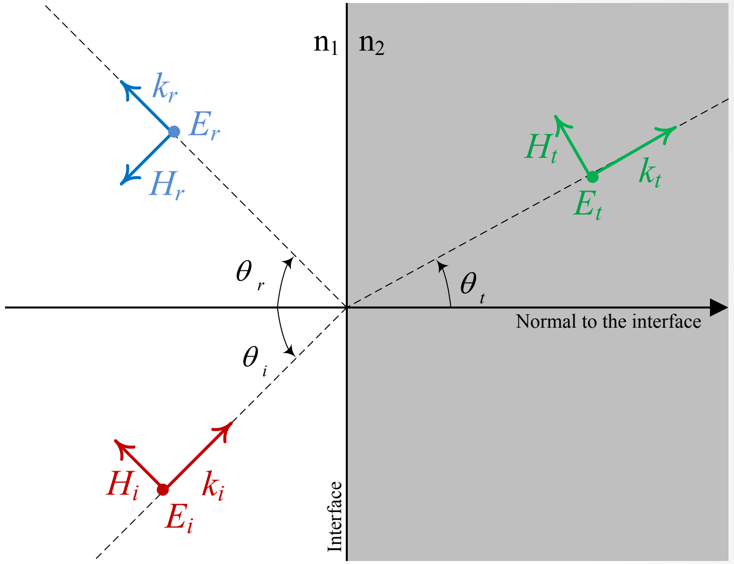
"plane" indicates the interface between the two medium.

"real³" indicates the propagation direction.

$\vdash_{def}$ is_valid_interface i =
let(n_1, n_2, p, n) = i in
$0 < n_1 \wedge 0 < n_2 \wedge$ plane p $\wedge$ is_normal_to_plane n p

t) $\wedge$

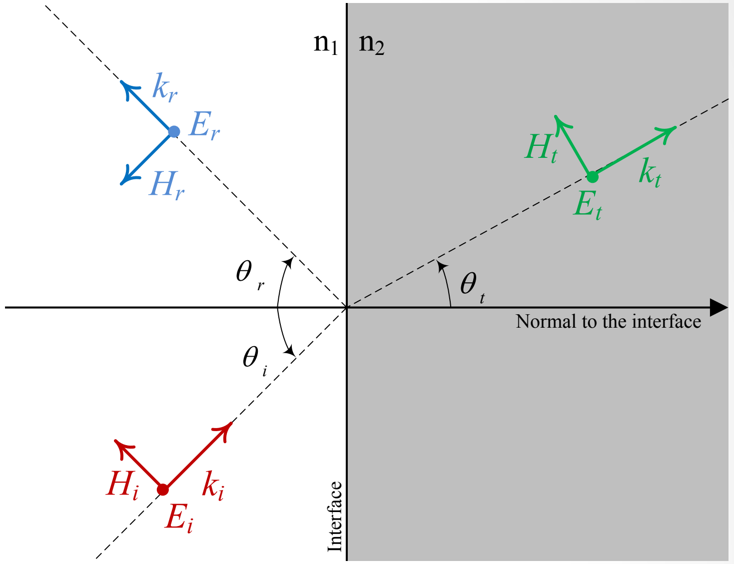
Formalization: Plane Interface



Physical Constraint: Plane wave at Plane Interface

$$
\begin{aligned}
\vdash_{def}\ & \mathtt{is\_plane\_wave\_at\_int\ i\ emf_i\ emf_r\ emf_t} \Leftrightarrow \\
& \mathtt{is\_valid\_interface\ i} \wedge \mathtt{non\_null\ emf_i} \wedge \\
& \mathtt{is\_plane\_wave\ emf_i} \wedge \mathtt{is\_plane\_wave\ emf_r} \wedge \mathtt{is\_plane\_wave\ emf_t} \wedge \\
& \mathtt{let}\ (n_1, n_2, p, n) = \mathtt{i\ in} \\
& \mathtt{let}\ (k_i, k_r, k_t) = \mathtt{map\_trpl\ k\_of\_w}\ (\mathtt{emf_i}, \mathtt{emf_r}, \mathtt{emf_t})\ \mathtt{in} \\
& \mathtt{let}\ (e_i, e_r, e_t) = \mathtt{map\_trpl}\ (\mathtt{norm} \circ \mathtt{e\_of\_w})\ (\mathtt{emf_i}, \mathtt{emf_r}, \mathtt{emf_t})\ \mathtt{in} \\
& \mathtt{let}\ (h_i, h_r, h_t) = \mathtt{map\_trpl}\ (\mathtt{norm} \circ \mathtt{k\_of\_w})\ (\mathtt{emf_i}, \mathtt{emf_r}, \mathtt{emf_t})\ \mathtt{in} \\
& 0 \leq (k_i \cdot n) \wedge (k_r \cdot n) \leq 0 \wedge 0 \leq (k_t \cdot n) \wedge \\
& (\forall pt.\ pt \in p \Rightarrow \forall t.\ \mathtt{boundary\_conditions}\ (\mathtt{emf_i} + \mathtt{emf_r})\ \mathtt{emf_t}\ n\ pt\ t) \wedge \\
& \exists k_0.\ \mathtt{norm}\ k_i = k_0 n_1 \wedge \mathtt{norm}\ k_r = k_0 n_1 \wedge \mathtt{norm}\ k_t = k_0 n_2 \wedge \\
& \exists \eta_0.\ h_i = e_i n_1/\eta_0 \wedge h_r = e_r n_1/\eta_0 \wedge h_t = e_t n_2/\eta_0
\end{aligned}
$$

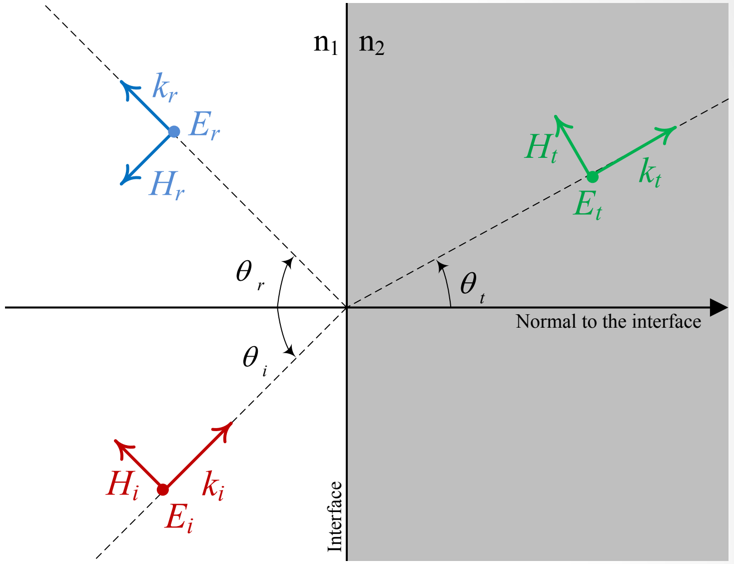
Formalization: Primitive Rules



- **Law of Reflection**

$\vdash \forall$i emf$_{\mathtt{i}}$ emf$_{\mathtt{r}}$ emf$_{\mathtt{t}}$.
is_plane_wave_at_int i emf$_{\mathtt{i}}$ emf$_{\mathtt{r}}$ emf$_{\mathtt{t}}$ $\wedge$
non_null emf$_{\mathtt{r}}$ $\Rightarrow$
are_sym_wrt ($-$(k_of_w emf$_{\mathtt{i}}$)) (k_of_w emf$_{\mathtt{r}}$)
(normal_of_plane (plane_of_interface i))

Formalization: Primitive Rules



- **Law of Reflection**
- **Snell's Law**

$\vdash \forall \mathtt{i}\ \mathtt{emf_i}\ \mathtt{emf_r}\ \mathtt{emf_t}.$

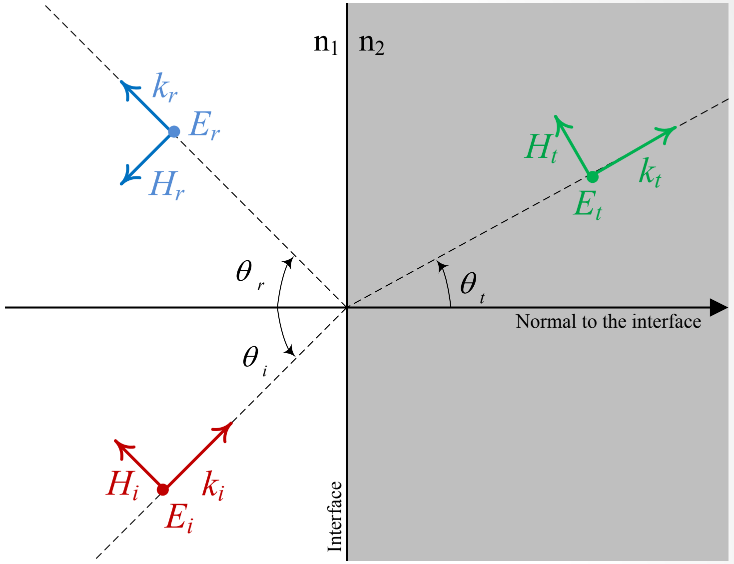
$\quad \mathtt{is\_plane\_wave\_at\_int}\ \mathtt{i}\ \mathtt{emf_i}\ \mathtt{emf_r}\ \mathtt{emf_t} \wedge$

$\quad \mathtt{non\_null}\ \mathtt{emf_t} \Rightarrow$

$\quad \mathtt{let}\ \theta = \lambda \mathtt{emf}.\ \mathtt{vectorangle}\ (\mathtt{k\_of\_w}\ \mathtt{emf})\ (\mathtt{normal\_of\_interface}\ \mathtt{i})\ \mathtt{in}$

$\quad \mathtt{n_1} \sin(\theta\ \mathtt{emf_i}) = \mathtt{n_2} \sin(\theta\ \mathtt{emf_t})$

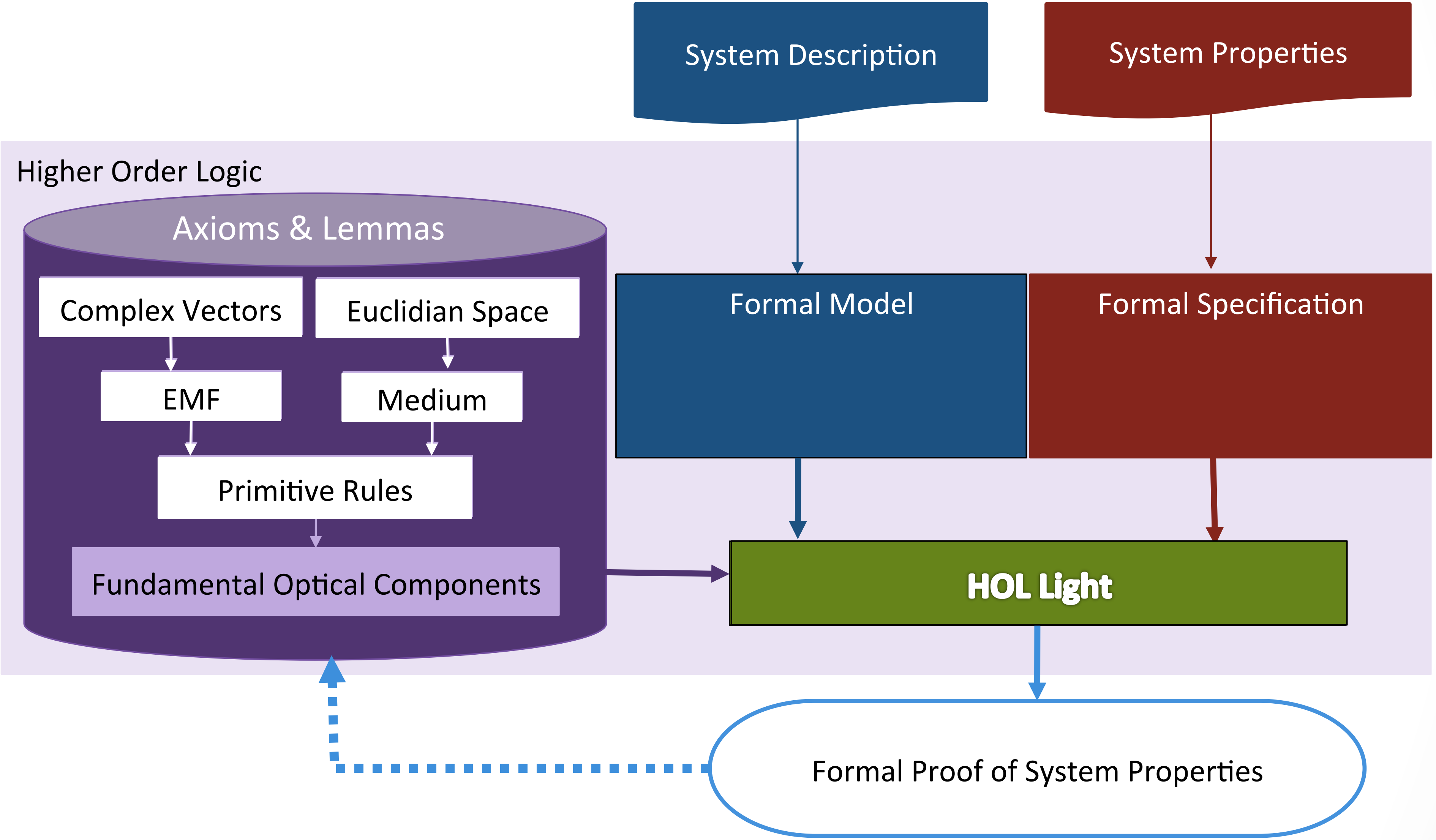
Formalization: Primitive Rules



- **Law of Reflection**
- **Snell's Law**
- **Fresnel Equations in TE mode**

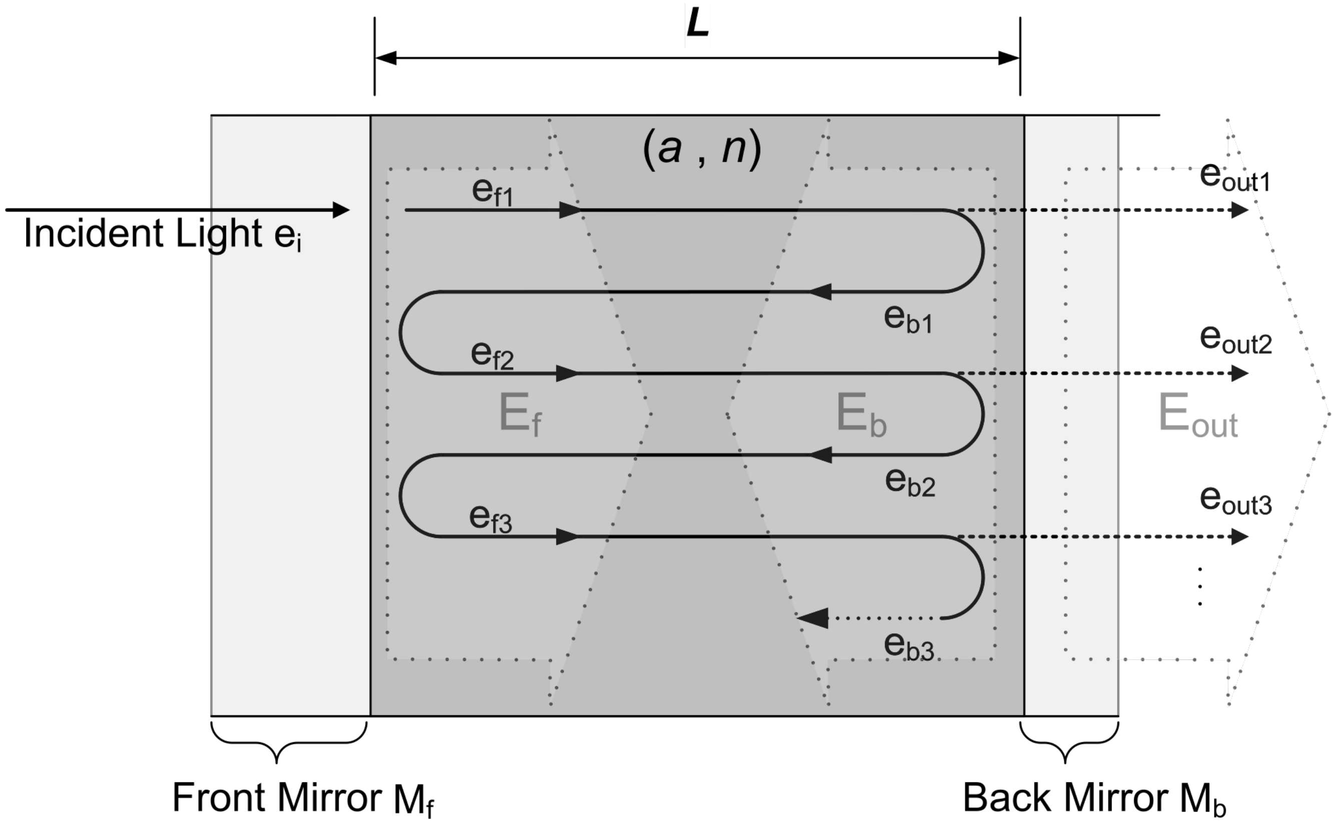
$$\vdash \forall \mathtt{i}\ \mathtt{emf_i}\ \mathtt{emf_r}\ \mathtt{emf_t}.$$
$$\mathtt{is\_plane\_wave\_at\_int}\ \mathtt{i}\ \mathtt{emf_i}\ \mathtt{emf_r}\ \mathtt{emf_t}\ \wedge$$
$$\mathtt{non\_null}\ \mathtt{emf_r}\ \wedge\ \mathtt{non\_null}\ \mathtt{emf_t}$$
$$\wedge\ \mathtt{te\_mode}\ \mathtt{i}\ \mathtt{emf_i}\ \mathtt{emf_r}\ \mathtt{emf_t} \Rightarrow$$
$$\mathtt{mag}\ \mathtt{emf_r} = \frac{\mathtt{n_1}\ \mathtt{ccos}(\theta\ \mathtt{emf_i}) - \mathtt{n_2}\ \mathtt{ccos}(\theta\ \mathtt{emf_t})}{\mathtt{n_1}\ \mathtt{ccos}(\theta\ \mathtt{emf_i}) + \mathtt{n_2}\ \mathtt{ccos}(\theta\ \mathtt{emf_t})}\ \mathtt{mag}\ \mathtt{emf_i}\ \wedge$$
$$\mathtt{mag}\ \mathtt{emf_t} = \frac{2\mathtt{n_1}\ \mathtt{ccos}(\theta\ \mathtt{emf_i})}{\mathtt{n_1}\ \mathtt{ccos}(\theta\ \mathtt{emf_i}) + \mathtt{n_2}\ \mathtt{ccos}(\theta\ \mathtt{emf_t})}\ \mathtt{mag}\ \mathtt{emf_i}$$

Towards Formalization of Optical Systems



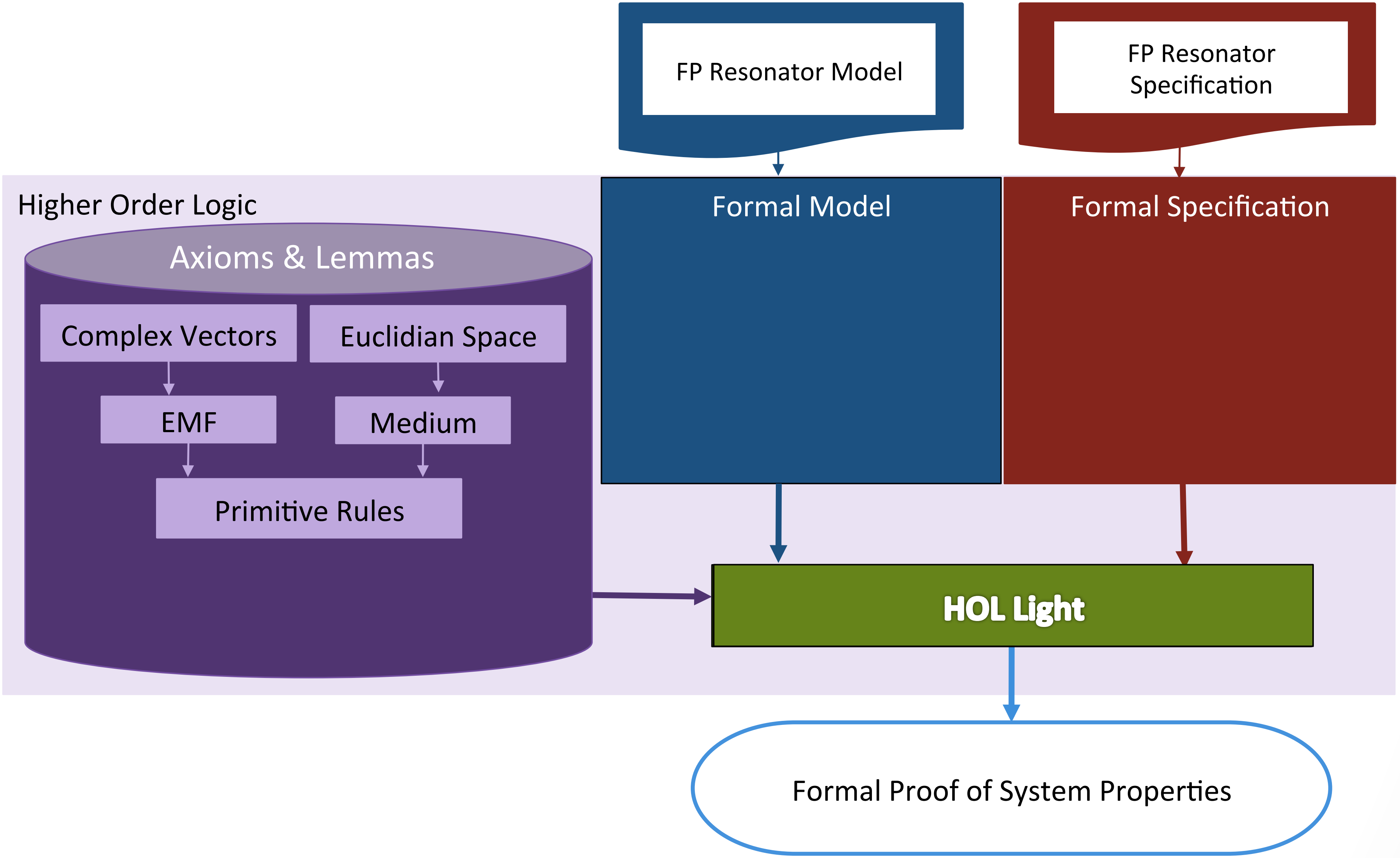
Fabry-Perot Resonator Structure

Two parallel partially reflecting mirrors



Based on the concepts of Constructive Interference of electromagnetic fields

Formalization of Fabry Perot Resonator



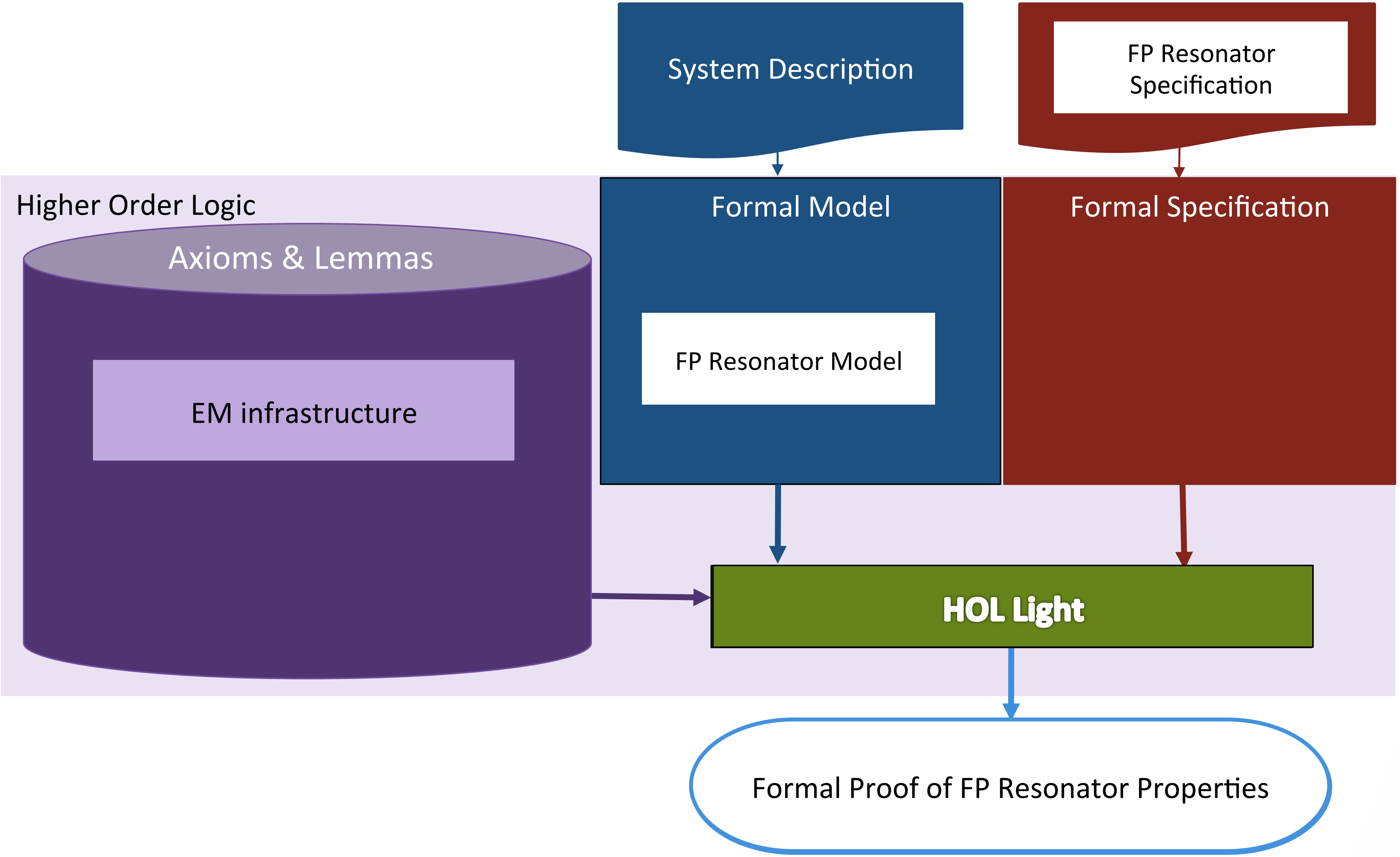
Formalization of Fabry Perot Resonator

$$\texttt{type fabry\_perot} = \texttt{interface} \times \texttt{interface} \times \texttt{real} \times \texttt{real}$$

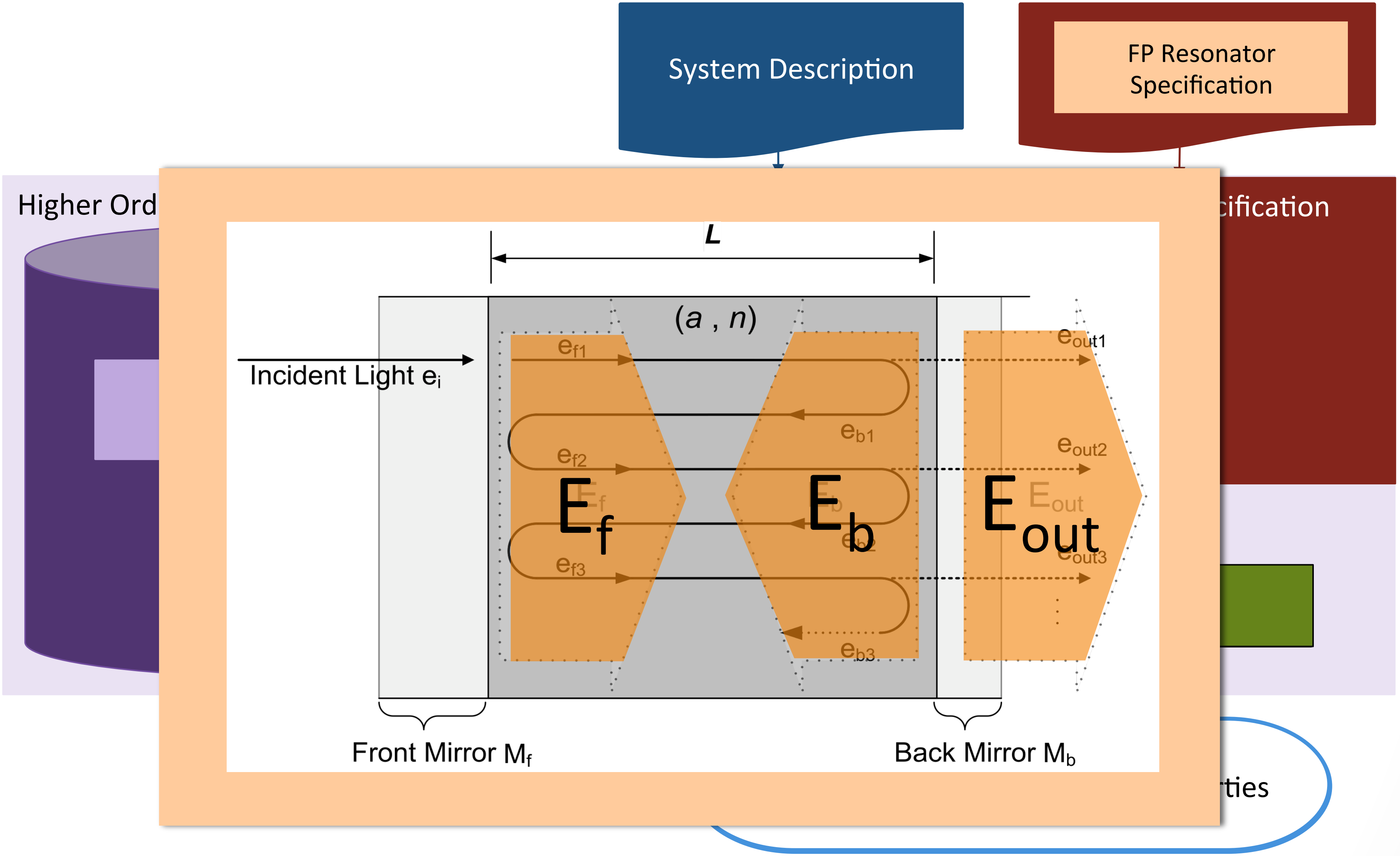
Physical Constraint: Valid Fabry Perot Resonator

$$
\begin{aligned}
\vdash_{\texttt{def}}\quad & \texttt{is\_valid\_fp fp} \Leftrightarrow \\
& \texttt{let } (\mathrm{M_f}, \mathrm{M_b}, \mathrm{a}, \mathrm{L}) = \texttt{fp in} \\
& \texttt{is\_valid\_interface } \mathrm{M_f} \ \wedge \texttt{is\_valid\_interface } \mathrm{M_b} \ \wedge \\
& \texttt{n}_2\texttt{\_of\_interface } \mathrm{M_f} = \texttt{n}_1\texttt{\_of\_interface } \mathrm{M_b} \ \wedge \\
& 0 < \mathrm{a} \ \wedge \ 0 < \mathrm{L} \ \wedge \\
& (\exists \alpha. \ \ (0 < \alpha \ \wedge \ \texttt{normal\_of\_interface } \mathrm{M_f} = \alpha \ \% \ \texttt{normal\_of\_interface } \mathrm{M_b})
\end{aligned}
$$

Formalization of Fabry Perot Resonator



Formalization of Fabry Perot Resonator



Formalization of Fabry Perot Resonator

System Description

FP Resonator Specification

$\vdash \forall$ emf fp.

is_valid_FP M_f M_b a l $\wedge$ is_plane_wave_at_int M_f emf_i emf_r emf_t $\wedge$

tm_mode M_f emf_i emf_r emf_t $\Rightarrow$

$$\text{let } (r_f, t_f) = \left(\frac{n_2 \cos\theta_1 - n_1 \cos\theta_2}{n_1 \cos\theta_2 + n_2 \cos\theta_1}, \frac{2n_1 \cos\theta_1}{n_1 \cos\theta_2 + n_2 \cos\theta_1}\right) \text{ in}$$

$$\text{let } (r_b, t_b) = \left(\frac{n_3 \cos\theta_2 - n_2 \cos\theta_3}{n_2 \cos\theta_3 + n_3 \cos\theta_2}, \frac{2n_2 \cos\theta_2}{n_2 \cos\theta_3 + n_3 \cos\theta_2}\right) \text{ in}$$

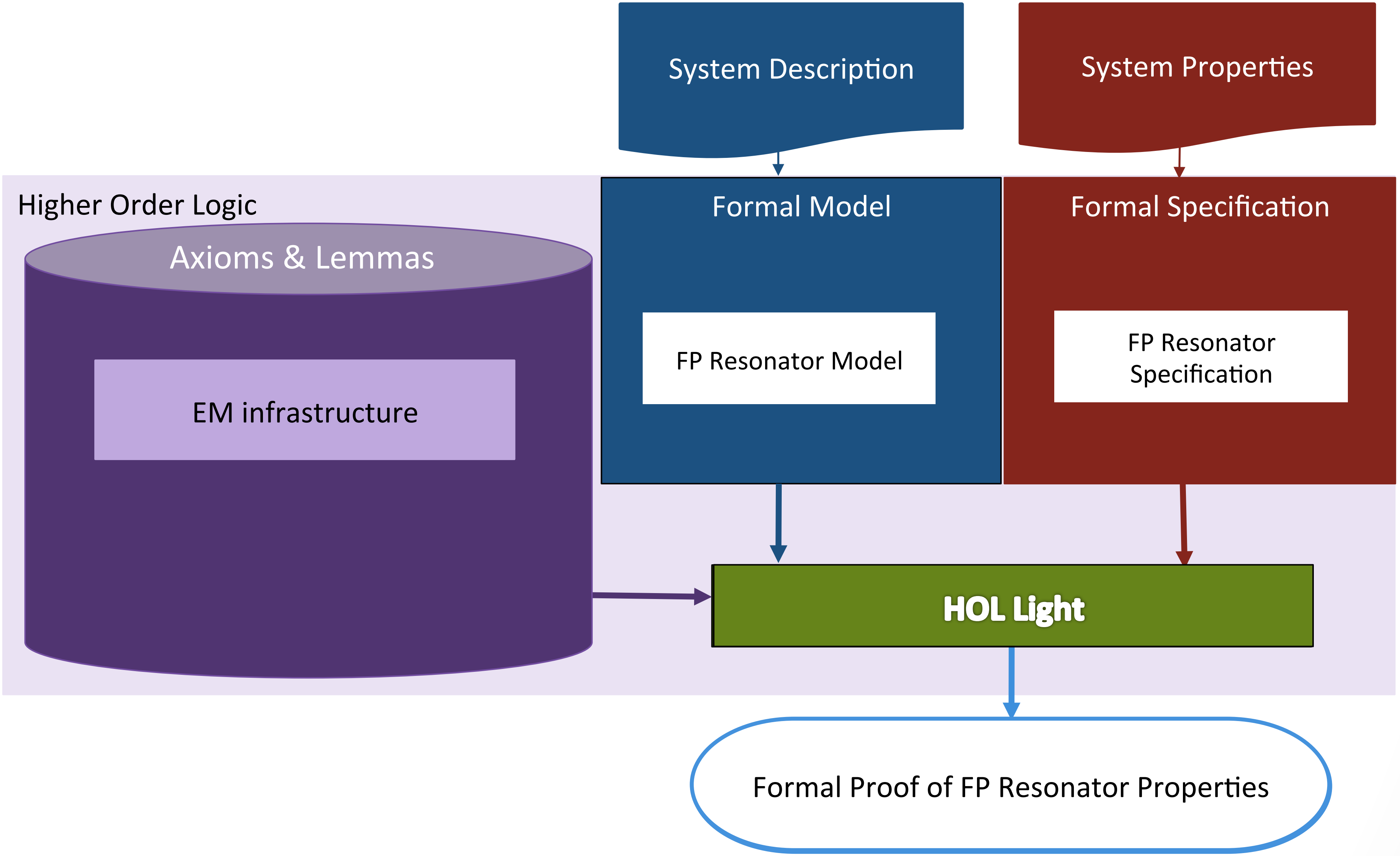
let mag = λemf. (TM_axis i emf_i emf_r emf_t $\cdot$ FST(mag_at_pln p n emf)) in

$$\text{mag } (E_f \text{ } emf_i \text{ fp}) = \frac{t_f}{1 - r_f r_b e^{-(a+2jk)l}} \text{ mag } emf_i \wedge$$

$$\text{mag } (E_b \text{ } emf_i \text{ fp} = r_b e^{-(\frac{a}{2}+jk)l} \text{ mag } (E_f \text{ } emf_i \text{ fp}) \wedge$$

$$\text{mag } (E_{out} \text{ } emf_i \text{ fp}) = \frac{t_f t_b e^{-(\frac{a}{2}+jk)l}}{1 - r_f r_b e^{-(a+2jk)l}} \text{ mag } emf_i$$

Formalization of Fabry Perot Resonator



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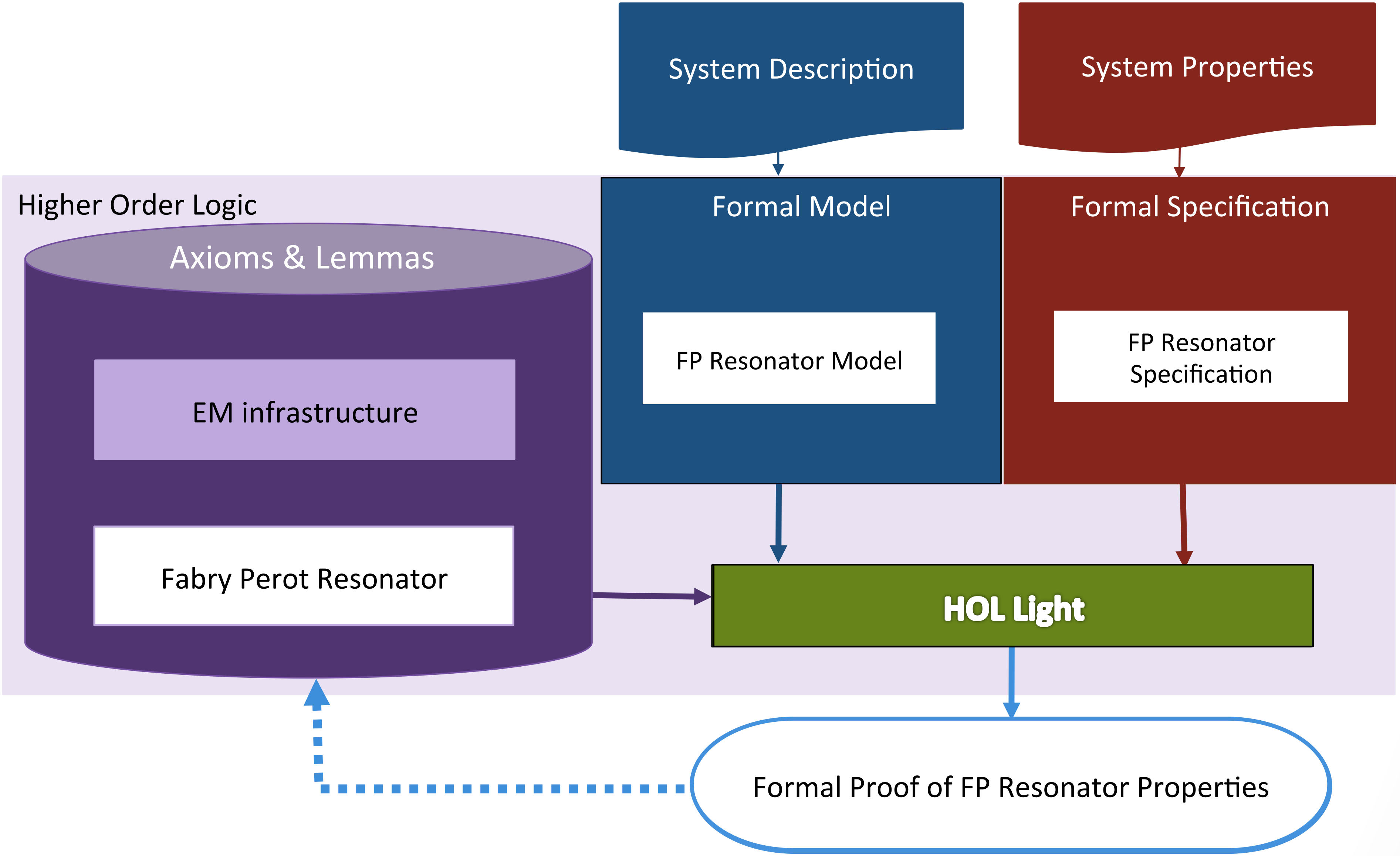
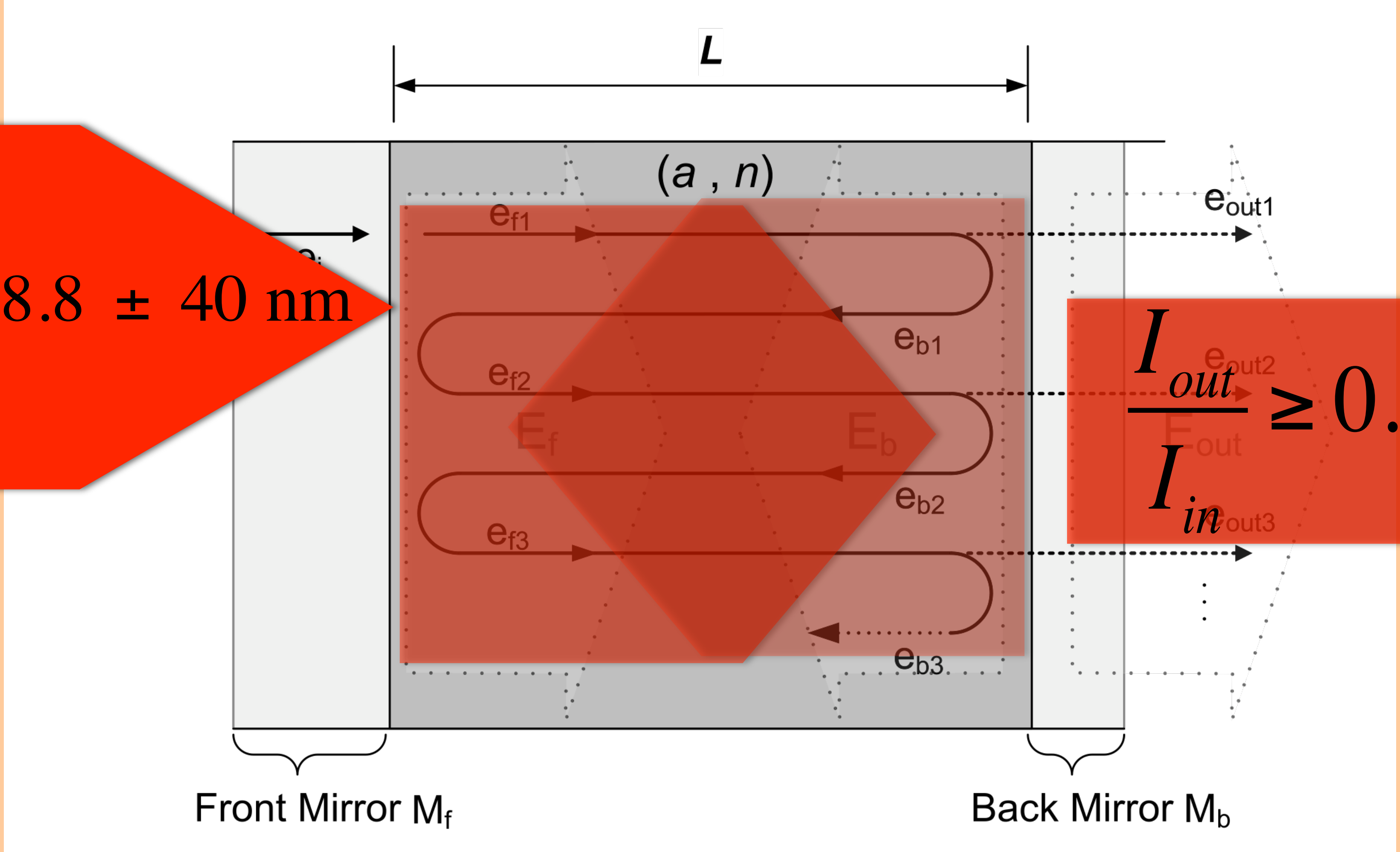


Case study: Intensity Ratio of a Fabry Perot Resonator

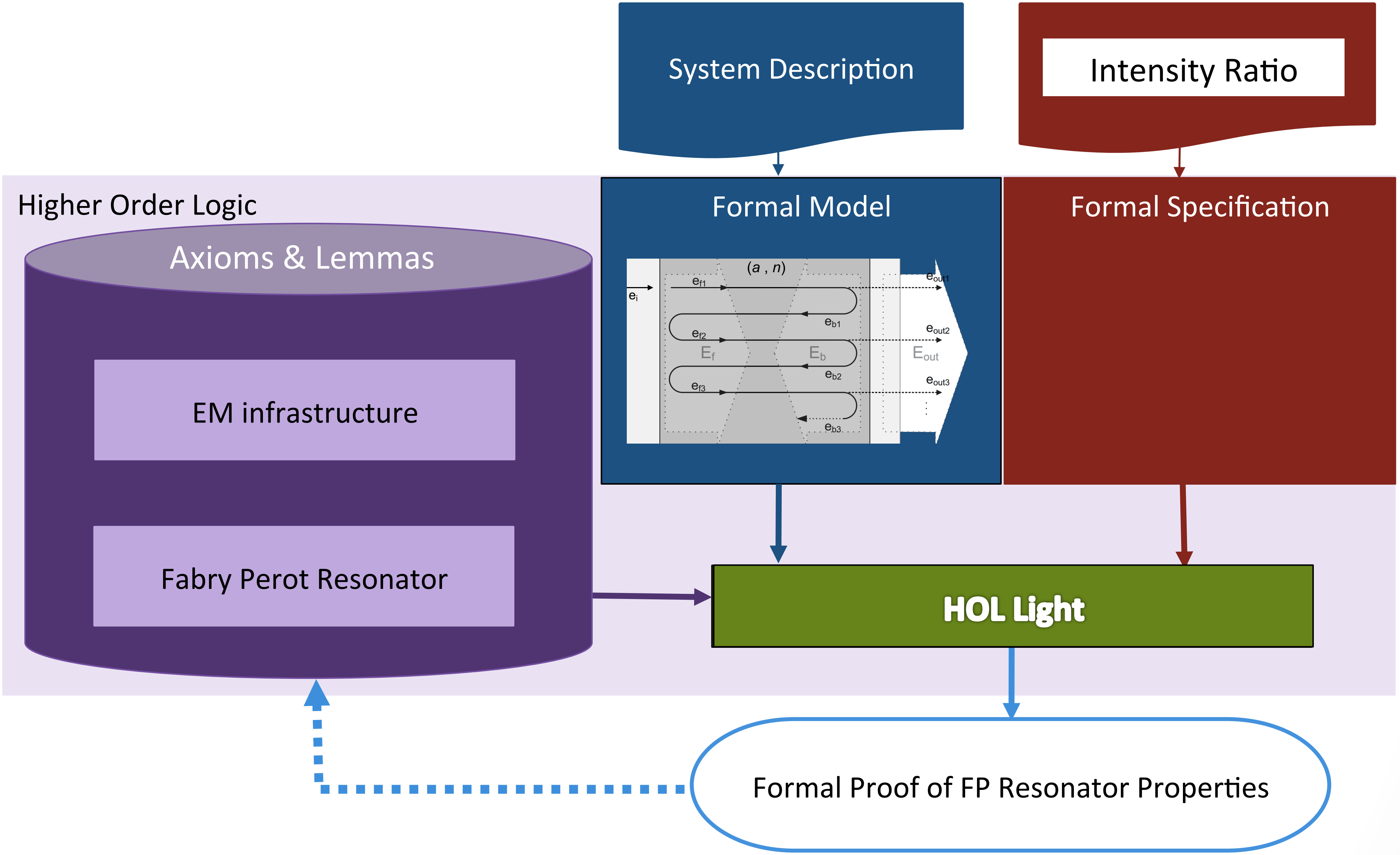
$$t_f = t_b = \sqrt{0.1} \;\wedge\; r_f = r_b = \sqrt{0.9} \;\wedge\; e^{-aL} = 0.98$$



$$\lambda = 638.8 \pm 40 \text{ nm}$$

$$\frac{I_{out}}{I_{in}} \geq 0.7$$

Formalization of Intensity Ratio of FP Resonator



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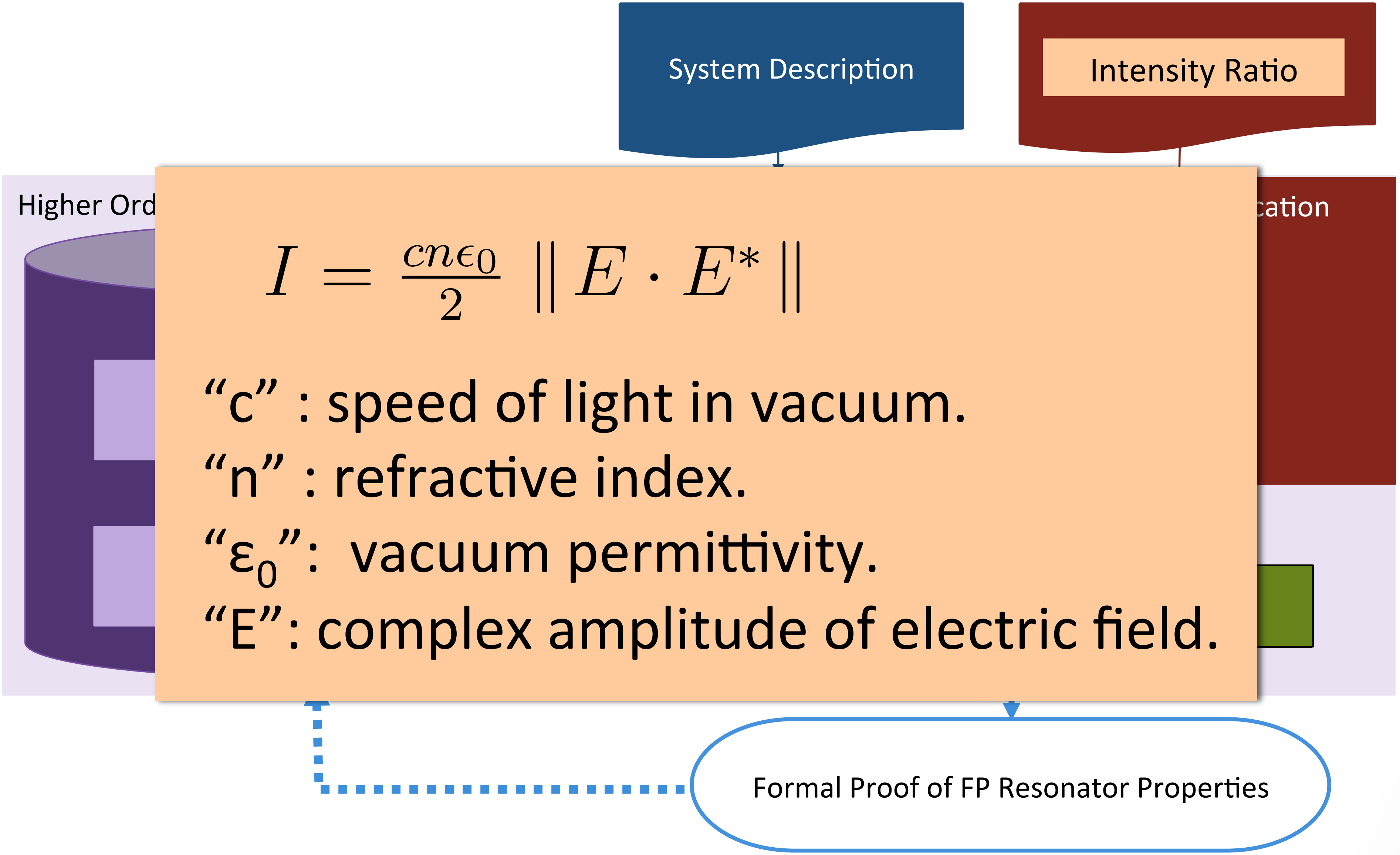


$$I = \frac{cn\epsilon_0}{2} \,\| E \cdot E^* \|$$

"c" : speed of light in vacuum.

"n" : refractive index.

"ε_0": vacuum permittivity.

"E": complex amplitude of electric field.

Formalization of Intensity Ratio of FP Resonator

System Description

Intensity Ratio

Intensity ratio:

$\vdash \forall$ emf fp.

is_valid_FP M_f M_b a l $\wedge$ is_plane_wave_at_int M_f emf_i emf_r emf_t $\wedge$

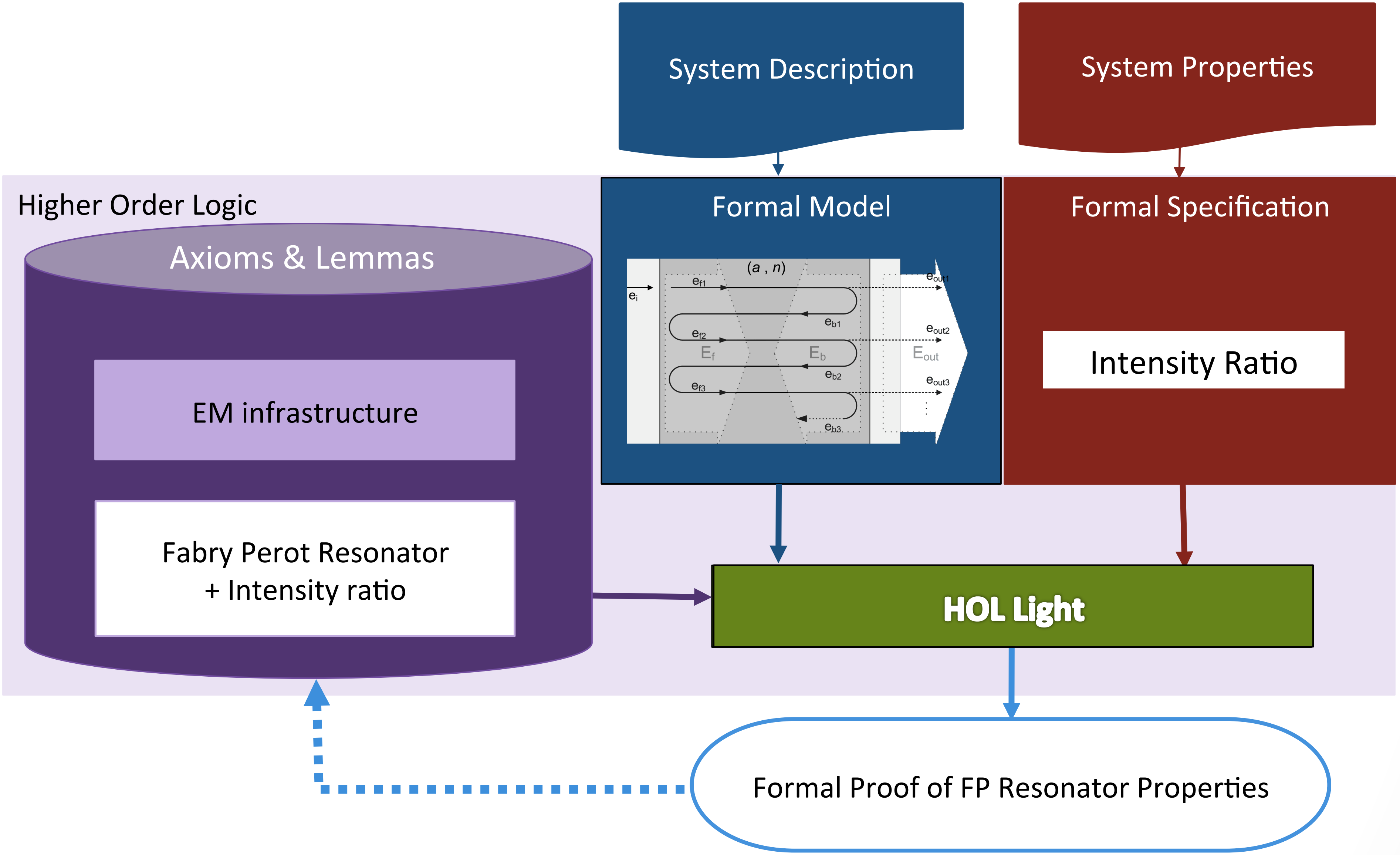
tm_mode M_f emf_i emf_r emf_t $\wedge$

n_1_of_interface M_f = n_2_of_interface M_b $\Rightarrow$

$$\text{intensity\_ratio emf fp} = \frac{t_f^2 t_b^2 e^{-al}}{(1 - r_f r_b e^{-al})^2 \left(1 + \frac{4 r_f r_b e^{-al}}{(1 - r_f r_b e^{-al})^2} \sin^2(kl)\right)}$$

Formal Proof of FP Resonator Properties

Formalization of Intensity Ratio of FP Resonator



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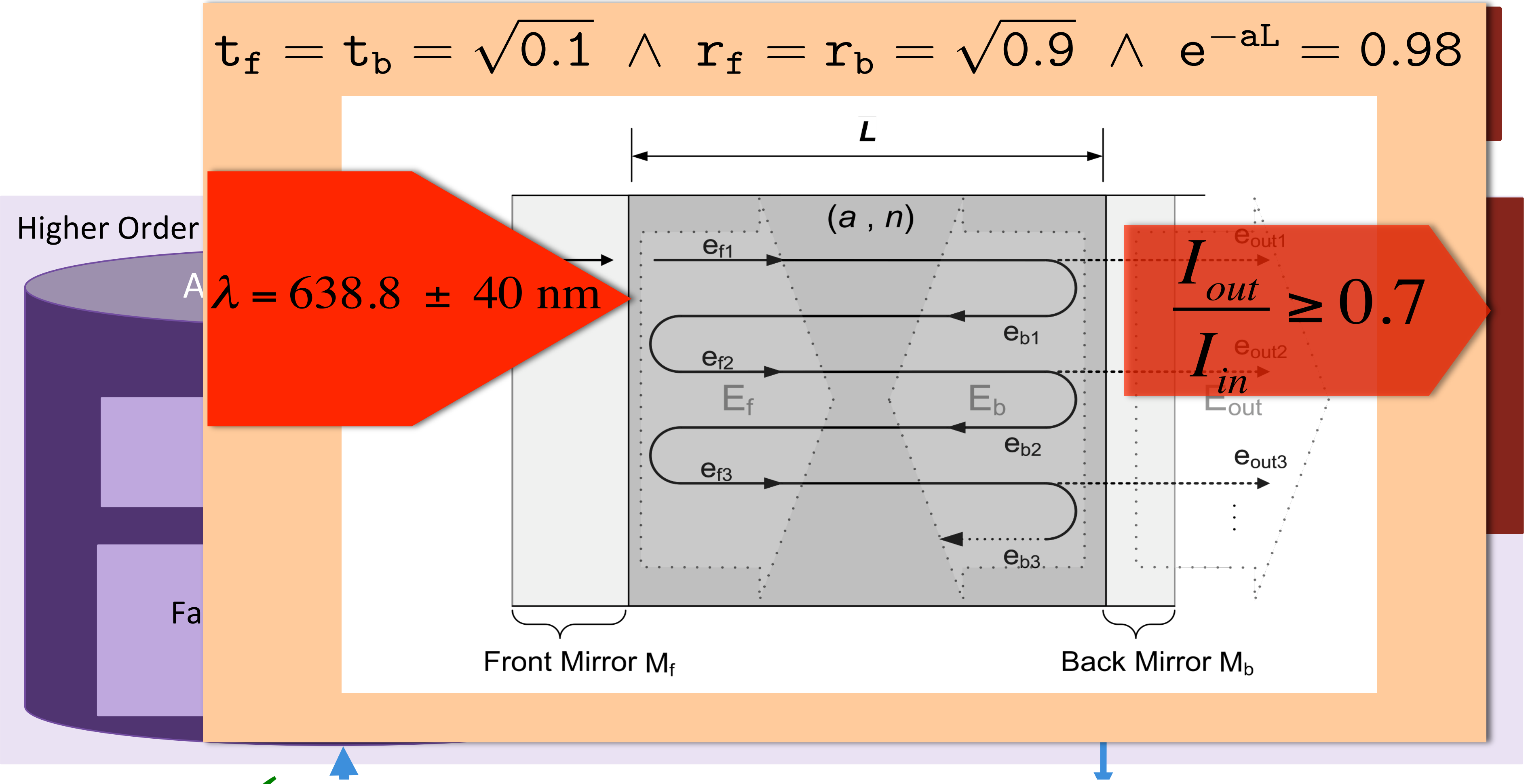


✓ **System Model ⇨ System Specification**

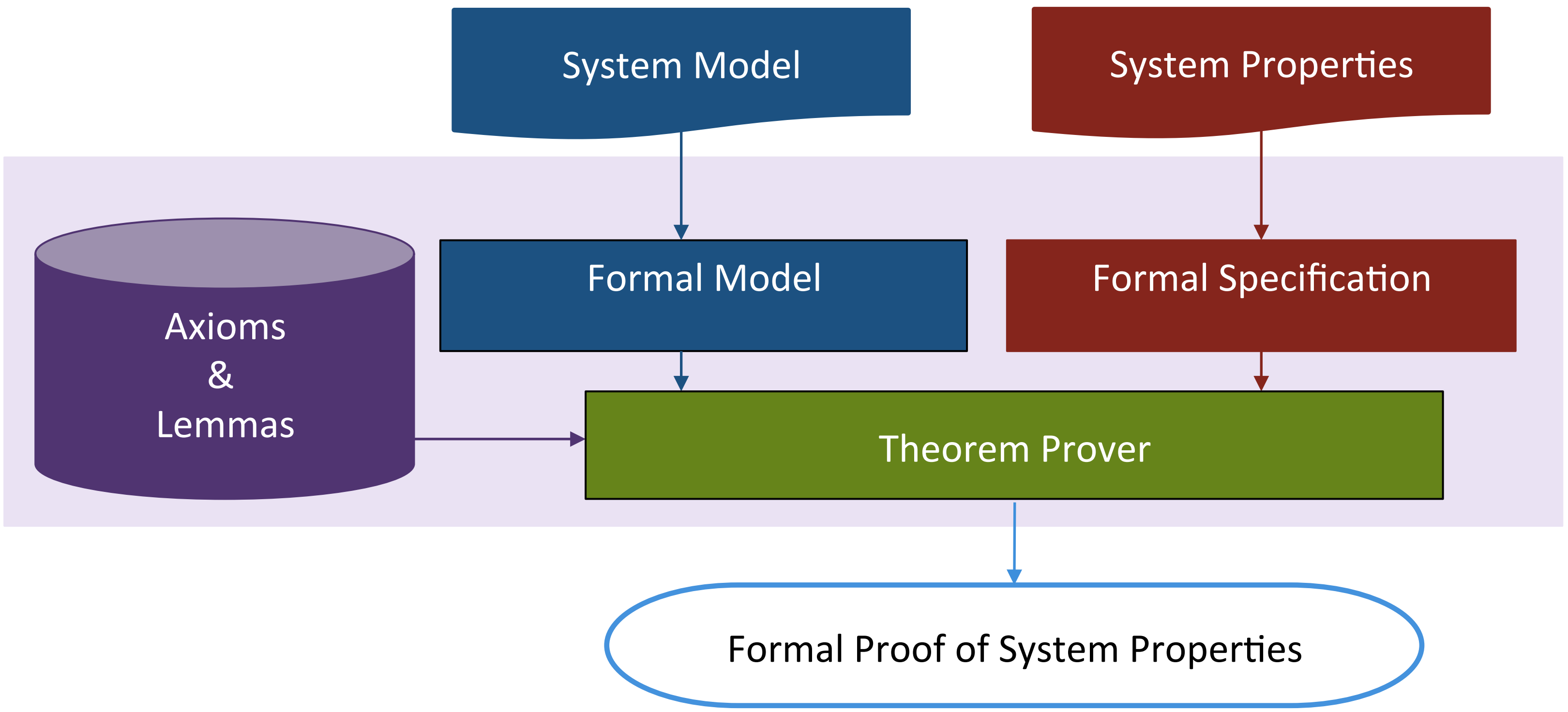
Conclusion

- A framework to formalize electromagnetic (and ray) optics
- Formalization of the infrastructures and fundamentals
- Many components can be addressed

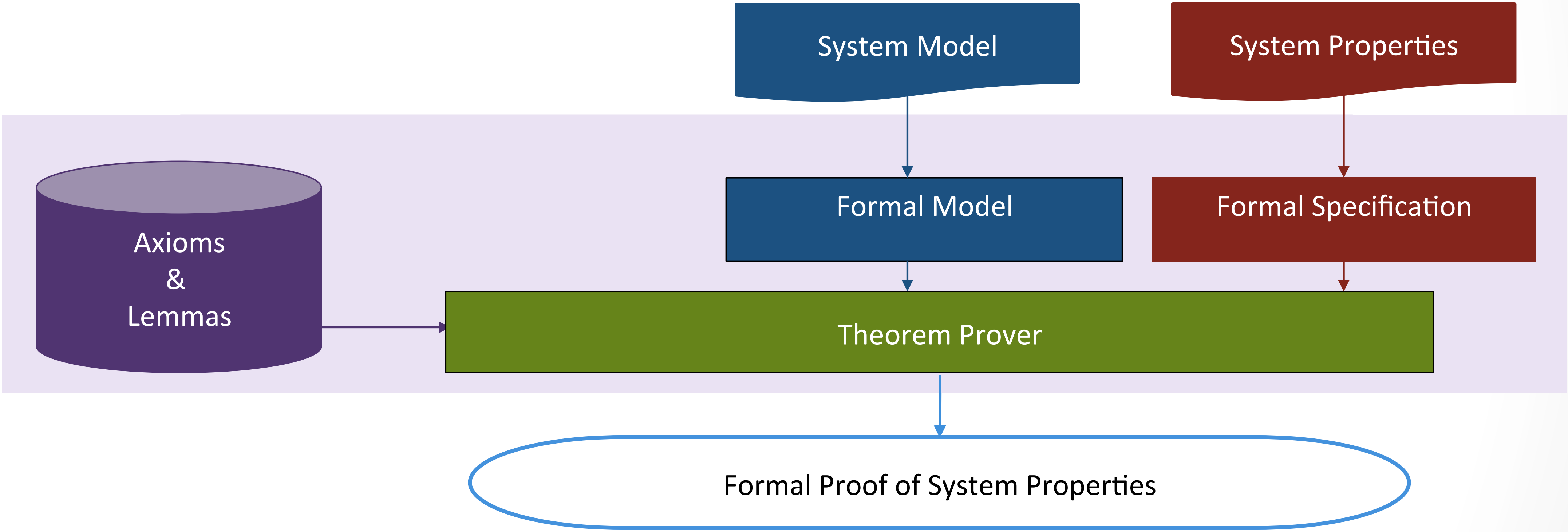
Future Work

- Enrich the libraries of optics
- Make a connection between our approach and traditional approaches

Extras in Formalization of Physics



Extras in Formalization of Physics



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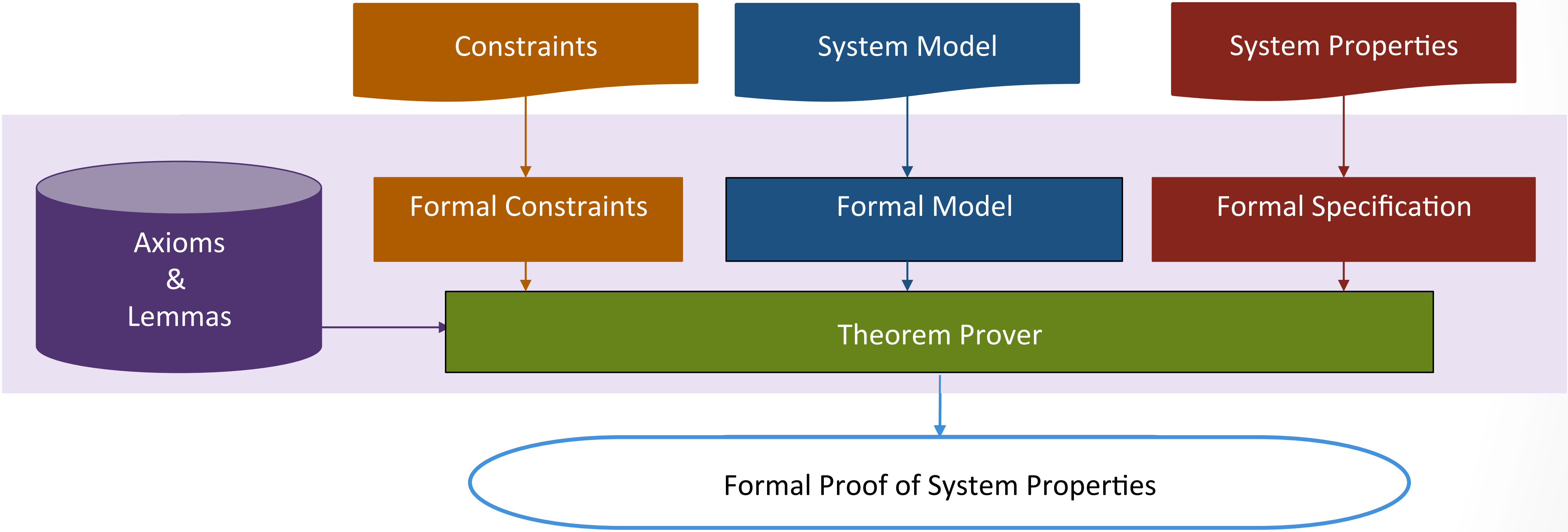

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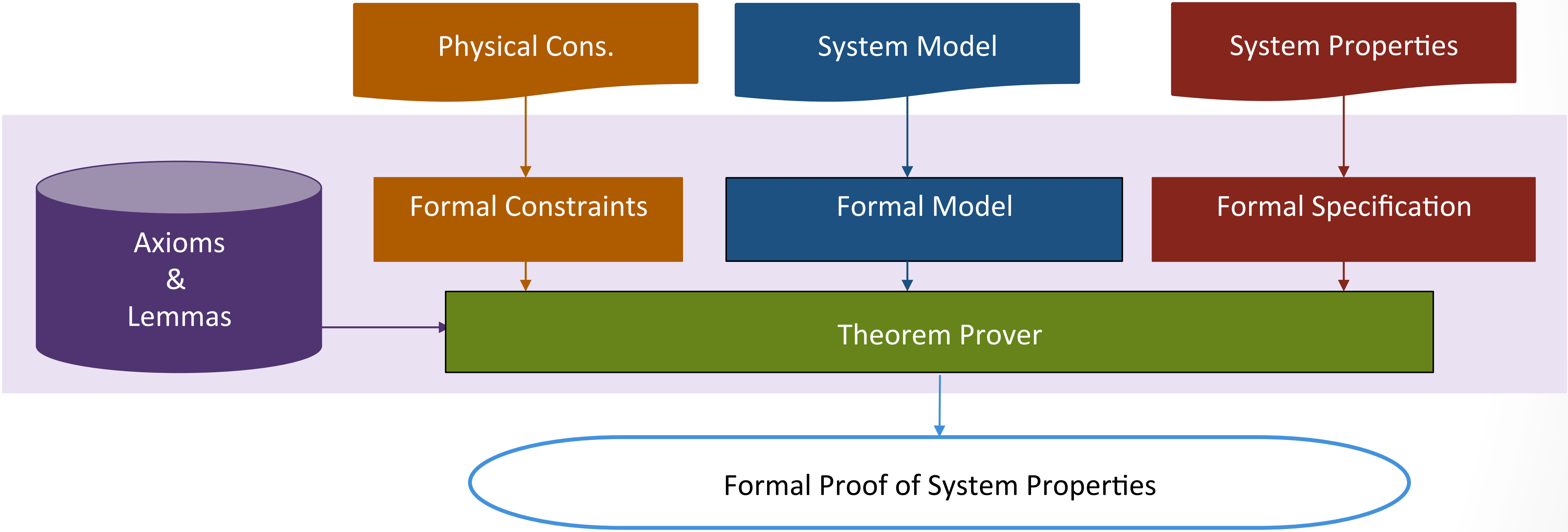

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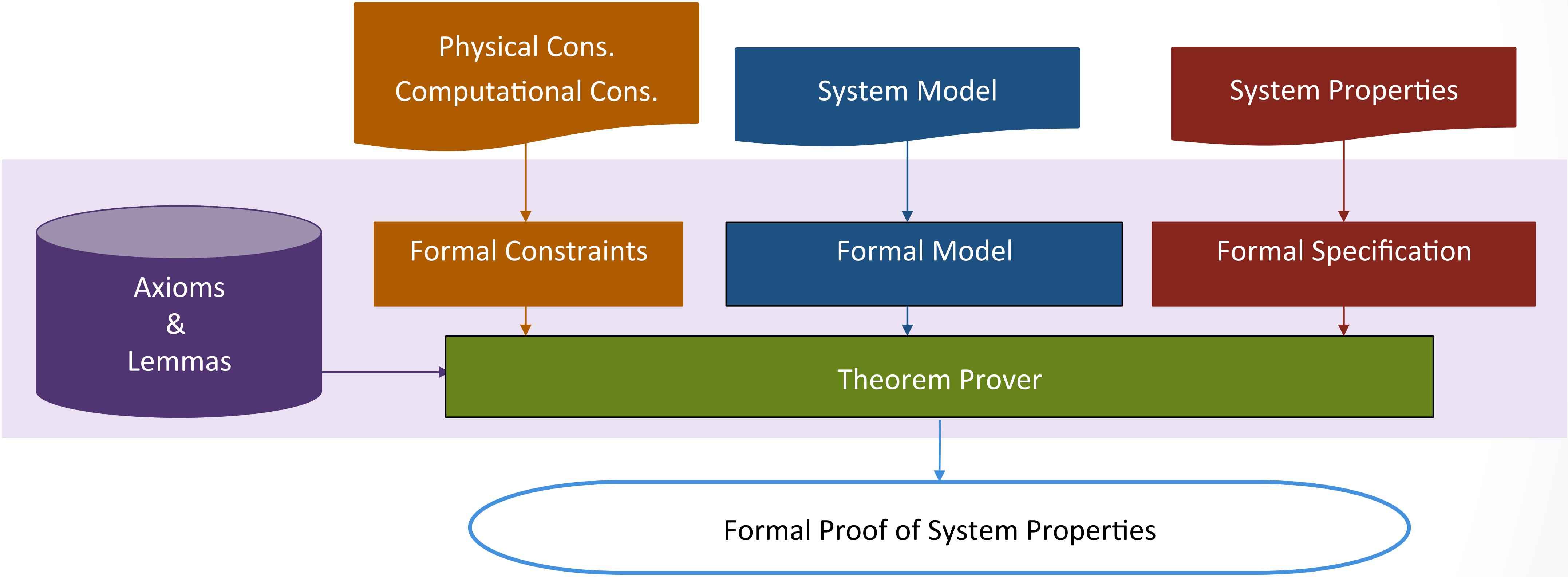

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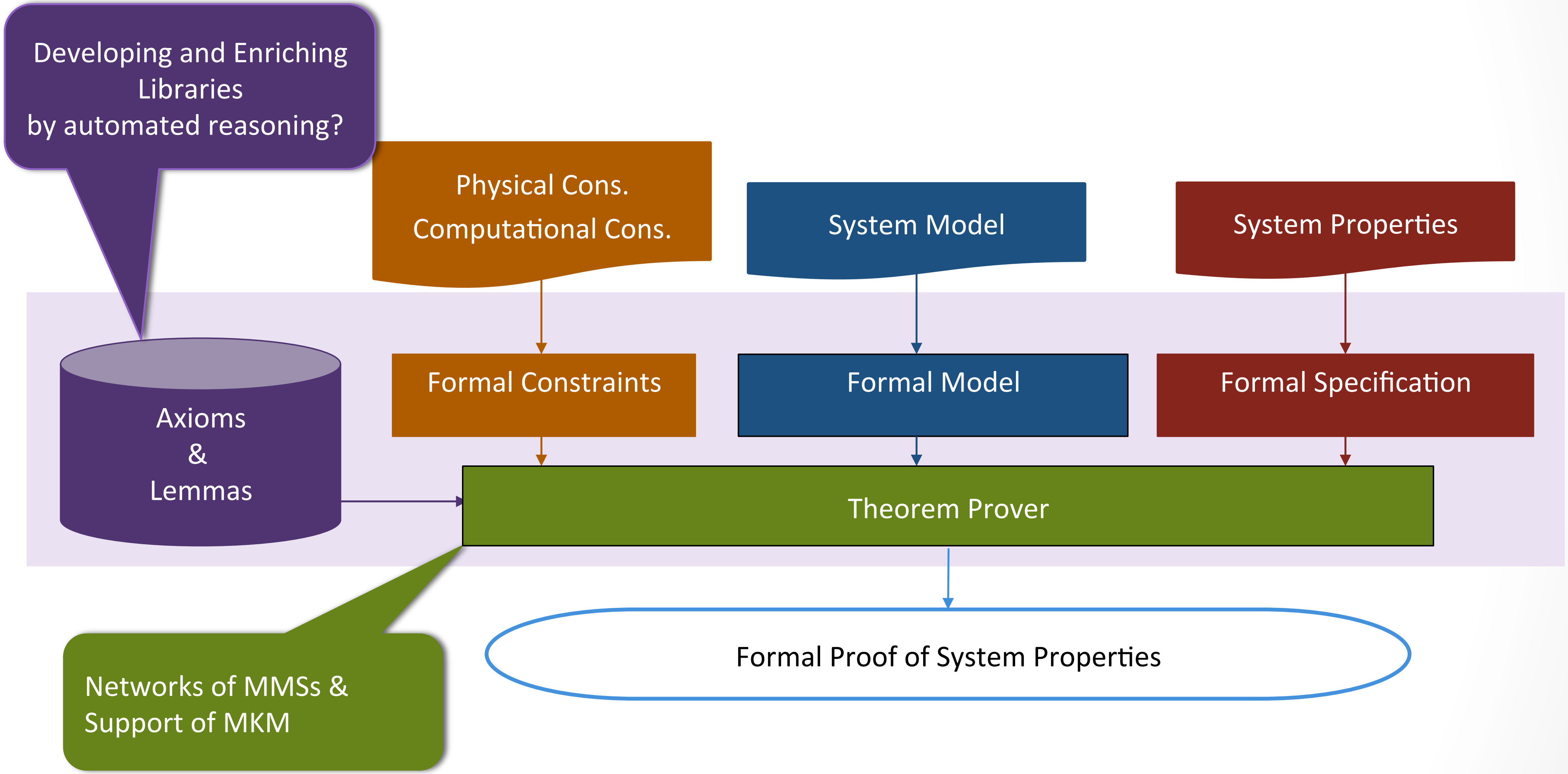

Concordia University

Hardware Verification Group

Faculty of Engineering and Computer Science

THANK YOU!

Formalization of Optics:
hvg.ece.concordia.ca/projects/optics/