Formalization of Partial Differential Equations using HOL Theorem Proving

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ABSTRACT

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Partial Differential Equations (PDEs) are central for the mathematical modeling of many physical and engineering problems such as heat transfer, the flow of fluids and the radiation of electromagnetic waves. Solving these equations is essential for gaining precise insights into the behavior of such systems. Traditionally, the analysis of PDEs has been performed using paper-and-pencil based proofs or computer-based numerical methods. However, these analysis techniques compromise the soundness and accuracy of their results, especially in safety-critical systems, due to the risk of human errors and inherent incompleteness of numerical algorithms. To address these limitations, we propose to use formal methods, in particular higher-order logic (HOL) theorem proving, for analyzing PDEs. The main motivation of this choice is the highly expressive and sound nature of HOL, which can be used to effectively model most systems that can be expressed in a closed mathematical form.

In this thesis, we introduce a comprehensive framework for the formal analysis of mathematical models of physical systems described by PDEs using the interactive proof assistant HOL Light. In particular, we have developed formal libraries for the heat, Laplace, telegrapher's and wave equations. Each library includes the formalization of these PDEs, encompassing their formal definitions and the formal verification

of some classical properties as well as their analytical solutions. These libraries constitute distinct contributions, each providing substantial value for various applications. To demonstrate the practical utility and effectiveness of our proposed framework, we conduct the formal analysis of several physical systems such as thermal protection, transmission lines, and potential flows.

In loving memory of my father and my aunt,

To my mother, my sister and my nephew.

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Anne, baba, Fevziye ve Alper, sizi çok seviyorum ve her daim özlüyorum.

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LIST OF ACRONYMS

CAS	Computer Algebra System
CTL	Coupled Transmission Line
FDM	Finite Difference Method
FEM	Finite Element Method
FVM	Finite Volume Method
HOL	Higher-Order Logic
IVP	Initial Value Problem
KCL	Kirchhoff's Current Law
KVL	Kirchhoff's Voltage Law
ML	Machine Learning
NS	Navier-Stokes
OCaml	Objective Categorical Abstract Machine Language
ODE	Ordinary Differential Equation
PDE	Partial Differential Equation
PVS	Prototype Verification System
RF	Radio-Frequency
STL	Signal Temporal Logic
WPT	Wireless Power Transfer

Chapter 1

Introduction

1.1 Motivation

Nothing puzzles me more than time and space; and yet nothing troubles me less, as I never think about them.

- Charles Lamb, *Letter to T. Manning (1810)*

The study of physical science has been instrumental in fostering the development of numerous important mathematical ideas. For instance, the origins of calculus can be traced back to the need to correctly describe the motion of bodies. Mathematics and physics have been closely connected, with mathematical equations playing an essential role in the formulation of many concepts in physics. However, the scope of this approach has been significantly expanded by mathematicians and scientists to encompass nearly all areas of science and technology, which has led to the emergence of a new field known as mathematical modeling in recent years [1]. A mathematical model is defined as an equation or a set of equations whose solutions characterize the physical behavior of a corresponding physical system [2]. Mathematical modeling requires a series of steps, beginning with physical observations and selecting the relevant physical variables. This is followed by the formulation of equations, the analysis of the equations, and simulation. Once these steps have been completed, the model must be validated. In this context, Partial Differential Equations (PDEs) [3] play a critical role for modeling dynamic forms in both mathematical and physical problems. Unlike algebraic expressions, PDEs involve first or higher-order derivatives of multivariate functions and aim to establish relationships between these functions and their derivatives. Consequently, the solutions must conform to these relationships, describing the variation of quantities with respect to multiple variables, typically space and time [4]. For instance, the flow of air past the wings of an airplane, the collapse of a star into a black hole, and the measurement of atmospheric pressure and temperature over different locations and times can be modeled by PDEs. There is a wide spectrum of different types of PDEs such as elliptic, hyperbolic, parabolic, linear and non-linear [3]. Notable examples of PDEs are the telegrapher's equations to represent the behavior of the voltage and current along a transmission line [5], the Laplace equation to analyze fluid flow problems [6], the wave equation to describe wave propagation [7], and the heat equation to model various processes involving diffusion of heat [8].

The study of PDEs can be divided into two main areas, namely, theoretical study and the construction of solutions for practical applications. The theoretical study of PDEs is a branch of pure mathematics, rooted in the foundational contributions of Lagrange, Euler, Bernoulli, Laplace and other pioneering figures from the early era of modern science [9]. Theoretical approach involves an understanding of the fundamental properties of PDEs, such as existence and uniqueness theorems, regularity theory, and the behavior of solutions, e.g., stability, asymptotic behavior, and singularity formation. On the other hand, the formulation of solutions for practical applications has required extensive modern mathematical and scientific research, resulting in the development of various numerical and analytical solution techniques. It is important to note that these two areas complement each other, as theoretical research guides the formulation of practical solutions and formulas, while practical applications drive new theoretical advancements in the study of PDEs.

Over the years, researchers have worked to develop both analytical and numerical techniques to solve PDEs that govern important problems such as aerodynamics [10], fusion plasmas [11] and climate modeling [12]. Numerical techniques represent a diverse range of approaches that employ computers to approximate solutions for certain PDEs. Among the most prominent numerical techniques are Finite Difference [13] and Finite Element Method [14]. The basic idea behind any numerical techniques to approximate solutions of PDEs is to replace the continuous problem with a discrete problem. Since exact solutions are either too complicated to determine in closed-form or are not known to exist in many cases, numerical techniques are highly effective in obtaining solutions to many difficult PDEs. However, the accuracy of the results is inherently constrained by the finite precision and spatial resolution of computer representations of the equations. Moreover, numerical solvers can be quite time-consuming and computationally expensive to run. In contrast to numerical techniques, analytical methods offer exact solutions to PDEs as they do not involve any approximation of the associated mathematical expressions. Therefore, they are preferred to numerical methods in order to guarantee the accuracy of the result. Furthermore, they provide a better understanding for the physical significance of numerous parameters that affect the problem. Some commonly used analytical techniques to solve PDEs are separation of variables [15] and transform methods [16]. The usage of separation of variables simplifies the process of finding solutions by transforming PDEs into Ordinary Differential Equations (ODEs) [17] which are simpler to solve [3]. Transform methods are also powerful approaches in solving the initial and boundary value problems for linear PDEs [18]. An initial value problem involves finding a solution to a PDE that satisfies specific conditions at the initial time, whereas a boundary value problem requires a solution that meets specified conditions along the boundaries of the spatial domain boundaries. These methods often yield closed-form solutions through the use of contour integrals [19].

Traditionally, PDEs have been analyzed using paper-and-pencil proof and computer-based numerical [20] and symbolic methods [21]. However, human error can occur in the former method because mathematicians are fallible and may overlook routine logical steps. Furthermore, not clearly stating all necessary assumptions in the analysis potentially leads to inaccurate results. Similarly, numerical and symbolic methods are based on approximation of the mathematical results due to the finite precision of computer arithmetic. Moreover, the core of the tools involved in the symbolic methods based analysis has a large number of unverified algorithms making the accuracy of the associated analysis questionable. Given the safety-critical nature of many systems, such as aerodynamics, telecommunications and transmission lines, these conventional techniques cannot ensure absolute accuracy of the analysis.

Unlike the above-mentioned approaches, formal methods [22] allow precise and accurate analysis, and can overcome the aforementioned limitations of traditional approaches. Formal methods are computer-based techniques used for the mathematical modeling, analysis, and verification of abstract and physical systems. They are applied using powerful software that mechanize classical mathematical reasoning processes. Two of the widely used formal verification methods are model checking [23] and theorem proving [24]. Model checking is an efficient and automatic verification technique for systems that can be expressed as finite-state machines. However, due to the complicated analysis of PDEs and expressibility requirements of mathematics needed for the analysis of PDEs, such as transcendental functions, limits, derivatives, integrals, etc., model checking is not suitable to analyze PDEs. On the other hand, theorem proving, in particular higher-order logic (HOL) theorem proving [25] is an interactive approach that can be used to analyze most systems, thanks to the high expressive nature of the underlying logic [26]. In fact, the expressiveness of HOL allows the description of most of the classical mathematical theories, including real numbers, multivariate calculus, higher transcendental functions and topological spaces. However, since HOL is neither complete nor decidable [27], proofs cannot be automated in general and hence HOL theorem proving requires human interaction with the computer software.

1.2 State-of-the-Art

Partial differential equations (PDEs) are examined through the analysis of their solutions. Due to the widespread use of PDEs in various fields of science and engineering, researchers employ a range of valuable techniques to analyze PDEs that model physical systems. This section provides an overview of these techniques.

1.2.1 Paper-and-Pencil based Proofs

The use of traditional paper-and-pencil proofs represents a fundamental technique and initial step in the construction of models of physical systems and their associated properties, based on an understanding of the underlying physical concepts. The behavior of the system is then analyzed by applying mathematical reasoning to the solution of PDEs using paper-and-pencil methods. A significant body of work exists in the literature that utilizes paper-and-pencil proof methods to analyze well-known PDEs such as the work in [28–30]. However, proofs done with paper-and-pencil methods have notable limitations. For instance, mathematicians may omit writing explicitly all necessary assumptions critical to the soundness of their analysis, particularly during long and complicated proofs that include numerous subcases or when dealing with large-scale systems.

1.2.2 Computers in Analyzing PDEs

In light of recent advancements in computer technology, scientists and engineers have increasingly used computers to solve mathematical equations and perform operations with improved efficiency. We now present various ways to use computers for PDE analysis:

1.2.2.1 Computer Simulation

The advent of computers led to a demand for numerical methods and their practical applications to offer reliable approximations for solving PDEs that model physical systems. Traditional approaches such as Finite Difference Method (FDM) [31], Finite Volume Method (FVM) [32], and Finite Element Method (FEM) [14], have been extensively developed and applied for this purpose over the years. A successful simulation of a physical system is dependent upon the utilization of both powerful computer hardware and mathematical algorithms. The former is necessary to facilitate the rapid calculation of complex numbers, while the latter is essential for accurately and efficiently solving the system of equations that represents the physical system. However, the presence of unverified symbolic algorithms, along with potential discretization and numerical errors, raises concerns about the accuracy of the analysis. Consequently, the potentially inaccurate results make computer-based simulation unreliable, which poses significant risks for safety-critical applications that rely on PDEs analysis. For instance, Alvarez [33] studied a combined approach of computer simulation and formal methods. The author employed Signal Temporal Logic (STL) which is a formal language used to describe properties of signals over time. The author's approach involves approximating the PDE with the FEM, which simplifies the problem to the control synthesis of a discrete-time linear system subject to regular STL constraints. While this work involves formal methods, it also includes numerical solution of the PDE. Converting the continuous PDE into a discrete-time linear system may lead to loss of information or introduce discretization errors. Therefore, the dynamics of the original PDE might not be perfectly captured in the discrete-time model, affecting the reliability of the control synthesis.

1.2.2.2 Computer Algebra Systems

Computer algebra is an interdisciplinary field that bridges mathematics and computer science focusing on the development, implementation, and application of algorithms designed for the manipulation and analysis of mathematical expressions [34]. Various computer algebra system (CAS) such as Mathematica [35], Maple [36] have been used to solve a variety of mathematical equations, particularly the applications relevant to partial differential equations (PDEs) in applied sciences and engineering. These systems are highly effective for computing mathematical solutions symbolically. However, they can be unreliable [37] due to the presence of unverified huge symbolic manipulation algorithms in its core, which are prone to contain errors. Given the importance and safety-critical nature of physical systems modeled by PDEs, this approach may not be ideally suited for their analysis.

1.2.2.3 Machine Learning Algorithms

In recent years, Machine Learning (ML) of differential equations has become a promising approach for discovering physical laws in complex systems [38]. The development of ML has introduced a novel approach to solve PDEs, driven by the growing availability of high-quality data from simulations and experiments [39]. In addition, the emerging field of scientific ML integrates numerical analysis with ML, and hence leverages advancements in both disciplines to tackle complex problems with enhanced precision. Moreover, ML algorithms can complement human mathematical abilities by automating certain tasks and assisting in mathematical exploration. This collaborative interaction between machine and human intelligence has the potential to not only advance the field of differential equations, but also to expand the possibilities for scientific discovery. Although these methods show considerable promise, most are purely data-driven and operate as black boxes, relying heavily on large datasets [40]. Moreover, the accuracy of the machine learning model heavily depends on the quality and representativeness of the training data. For instance, incomplete, erroneous or inappropriate training data can lead to inaccurate or unreliable solutions [41]. As a result, these limitations can present challenges, particularly in the context of safety-critical systems, where they may conflict with established safety protocols.

1.2.2.4 Proof Assistants

Proof assistants are powerful tools that use a formal language in order to create mathematical concepts. These mathematical concepts are expressed in an appropriate logic, which might be a propositional [42], first-order [43], or higher-order logic [26] depending on the requirement for expressibility. The decidability of the underlying logic determines whether theorem proving can be performed automatically or interactively.

In the case of decidable logics, such as propositional logic, automatic verification can be achieved through the use of computer programs. In contrast, undecidable logics, like higher-order logic, require interactive verification, where user involvement is essential to guide the process of proving theorems. Moreover, proof assistants provide assurance that a logical argument is sound and that all possible cases have been comprehensively considered. Consequently, they enable the formalization of classical mathematics and the formal analysis of physical systems modeled by PDEs, thereby reducing errors and enhancing the reliability of the proofs.

In the next section, we provide more details about the use of proof assistants for the formalization of mathematics.

1.3 The Formalization of Mathematics

The formalization of mathematics is writing mathematical concepts such as definitions, theorems and proofs in the language of the proof assistant or theorem prover, to make them suitable for computer processing. There are numerous computer proof assistants available. One of the pioneering proof system is the de Brujin's *Automath* project, which was introduced in the late 1960s [44]. The project played a pivotal role in the development of type systems called type theory, which later influenced many modern proof assistants [45]. For a computer to be able to perform the formal verification of a theorem, it is necessary for it to be able to deduce the theorem from the axioms of the foundational system employed by the proof assistant, which is typically based on set theory or type theory [46].

Many proof assistants such as HOL4 [47], HOL Light [48], Isabelle/HOL [49], Coq [50], Lean [51] and PVS [52] utilize type theory as their logical foundation. In this context, HOL4, HOL Light and Isabelle/HOL use simple type theory called higher-order logic, which is rooted in Church's type theory [53]. In the simple type theory, terms can be higher-order, meaning that they can represent higher-order values like sets, relations, and functions [54]. Also, types are constructed inductively, and variables are assigned fixed types [55]. On the other hand, Lean and Coq use dependent type theory, enabling both types and elements to be parametrized by elements of other types. This capability provides a higher level of expressiveness and flexibility. Other proof assistants such as Mizar [56] and Metamath [57] are based on set theory to develop mathematical library. Each proof assistants has its own unique features and strengths. The choice of a proof assistant often depends on several factors including the availability of libraries, automation tools, usability and learning curve as well as the strength of community and support. Among the above-mentioned proof assistants, Coq, Lean, Isabelle/HOL and HOL Light are the most popular ones.

A large number of mathematical proofs have been formalized using different proof assistants. For instance, the Prime number theorem has been formalized by Harrison [58] and Avigad [59] in HOL Light and Isabelle/HOL, respectively. Moreover, Gonthier [60] formally verified the proofs of the four colour theorem in Coq. Another important work is the proof of the Kepler conjecture that was formalized by Hales et al. [61] using HOL Light and Isabelle/HOL. Furthermore, Paulson [62] has formalized another significant theorem, namely, Gödel's incompleteness theorem in Isabelle/HOL. There are many other contributions in formalization of mathematics beside the aforementioned examples. Nonetheless, these examples demonstrate the power of proof assistants in mathematical proofs, while also revealing their potential to provide new insights into the proofs.

There are also additional advantages for using computer proof assistants beyond the development of formal verification of theorems. Despite the extensive availability of printed resources, building a formalized library enables reuse and systematic updates. In addition, proof assistants can be used for educational purposes. For instance, it might be possible for undergraduate students to verify the correctness of their proofs and identify any mistakes [46]. Moreover, the process of formalization can improve the understanding of students in mathematical subjects, even for those who are already familiar with them. Furthermore, proof assistants enable researchers from different fields to work together effectively, which in turn leads to novel ideas and innovations.

As proof assistants become increasingly prevalent in the formalization of mathematics, enhancing the precision, explicitness, and reliability of mathematical proofs, we have opted to use this technology in our research. In particular, we have chosen the HOL Light proof assistant for the analysis of physical systems governed by partial differential equations. The reason behind this choice is the availability of a rich set of libraries for multivariable calculus, such as differential, integration, transcendental and real analysis [63]. In the following section, we provide an overview of related work on differential equations based formal analysis in different proof assistants.

1.4 Related Work

There are a number of studies that focus on the formal analysis of engineering and physical systems using differential equations in various theorem provers. For instance, Immler et al. [64] formalized Euler's method to approximate solutions of Ordinary Differential Equations (ODEs) in Isabelle/HOL. The authors focused on Initial Value Problems (IVPs) of ODEs and provided a formal verification of the Picard-Lindelöf theorem, which asserts the existence of a unique solution to the ODE. Similarly, Immler et al. [65] presented formal reasoning support for the flow of ODEs using Isabelle/HOL. In particular, they provided the formal verification for an ODE solution with different initial conditions. They also formalized the Poincaré map and formally verified its differentiability. However, both approaches are based on approximating the solutions of the differential equations that characterize the dynamical behavior of the underlying system. Guan et al. [66] used the HOL Light theorem prover to formalize the Euler-Lagrange equation set that is based on Gâutex derivatives [67]. Furthermore, the authors applied their proposed formalization to the formal verification of the least resistance problem of gas flow. Similarly, Sanwal et al. [68] formally verified the solutions of the second-order homogeneous linear differential equations using the HOL4 theorem prover. Moreover, they used their proposed formalization to formally verify the damped harmonic oscillator and a second-order op-amp circuit. In another effort, Rashid et al. formalized the Laplace [69] and the Fourier [70] transforms using HOL Light and used these formalizations for differential equations based analysis of many systems, such as an automobile suspension system [70], an unmanned free-swimming submersible vehicle [71] and a platoon of automated vehicles [72]. However, existing formalizations of ODEs in HOL4 and HOL Light, do not cater for the formalization of the solutions when dealing with partial differential equations. More recently, Bobbin et al. [73] formalized chemical physics using the Lean theorem prover. They constructed a foundational framework for equations related to motion or thermodynamics such as kinematic equations of motion or gas laws. However, their constructed proof is limited to ODEs. Park et al. [74] formalized Taylor models and power series for solving ODEs. The authors expanded their formalization of exact real computation to encompass precise approximations of classical partial functions using polynomials, analytic functions, and solutions to initial value problems for nonlinear polynomial

ODEs. Despite their contributions, their study focuses on univariate functions and is constrained to the context of ODEs.

Boldo et al. [75] employed the Coq theorem prover to formally verify the numerical solution of one-dimensional acoustic wave equation. The authors used the second-order centered finite difference scheme, commonly known as the three-point scheme for convergence of the result. In [76], the same authors mechanically verified the correctness of a C program implementing a numerical scheme for the solution of PDE, using both automated and interactive theorem provers. While both of these works made significant contributions, they approximate the solutions of acoustic wave equations and do not provide the analytical solution. More recently, Otsuki et al. [77] formalized the method of separation of variables and superposition principle, and applied it to the analysis of a one-dimensional wave equation using the Mizar theorem prover. However, they did not extend the solution for infinite series. To the best of our knowledge, there is no work that tackles the formalization of PDEs used to model physical systems, such as the heat, Laplace and telegrapher's equations, by analyzing their analytical solutions using HOL theorem proving.

1.5 Proposed Methodology

The objective of this thesis is to develop a theorem proving based framework that can handle the analysis of PDEs used in real-world systems within the sound core of the HOL Light theorem prover [48]. We develop foundational libraries for the heat, Laplace, telegrapher's and wave equations. Each library involves formalizing the corresponding PDEs and the verification of their analytical solutions. This framework ensures precise modeling and analysis of physical systems.

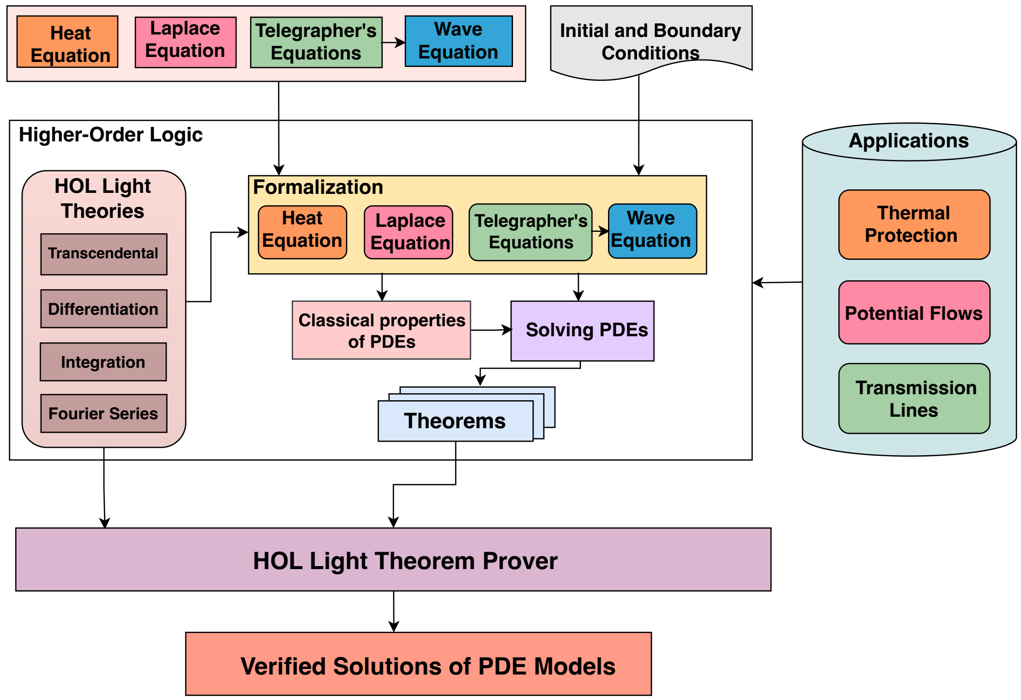


Figure 1.1: Proposed Methodology

The proposed methodology shown in Figure 1.1, outlines the main idea behind the theorem proving based analysis of systems involving PDEs. The inputs to this framework, depicted by a rectangle and the rectangle with curved bottom, are purely mathematical representations of PDEs. They model the dynamics of the underlying systems as PDEs and initial and boundary conditions used in solving these PDEs. The first step in the proposed methodology is to construct the corresponding model of the given PDE in higher-order logic. For this purpose, the foremost requirement is the ability to formalize these PDEs, in our case, the heat, Laplace, telegrapher's and wave equations as higher-order logic functions. The formalization of these PDEs requires the mathematical theories of real and complex numbers, transcendental, differentiation, integration which are available in respective HOL Light libraries as shown in Figure 1.1. The second step is to prove relevant some classical properties of the PDEs, such as linearity and homogeneity in HOL. Next, we have to choose appropriate solution techniques such as the method of separation of variables, transform methods or another suitable technique, in solving these PDEs analytically. The next step for conducting the formal analysis of the PDEs in a theorem prover is to formally verify the theorems developed in the previous steps by using pre-verified theorems corresponding to some commonly used properties of PDEs, such as linearity, differentiability, and summability. Finally, the output of the theorem proving based framework for PDE analysis provides the verified solutions of the PDE models. We illustrate the effectiveness of our proposed framework by conducting a formal analysis of PDEs that model physical systems, including thermal protection, potential flows, and transmission lines as depicted in Figure 1.1. This analysis enables a comprehensive understanding of the dynamic behaviors of these systems through their solutions. In particular, we formally analyze the heat transfer in one-dimensional rectangular slab using the heat equation [8]. Moreover, we formally model the Laplace equation, whose solutions are harmonic functions that play a crucial role in potential flow theory and, by extension, in aerodynamics. For instance, we conduct the formal analysis of the Rankine oval, potential flow past a circular cylinder, and potential flow past a rotating circular cylinder [78]. Finally, we formally model the telegrapher's equations, also referred to as the transmission line equations, along with their alternative representation in the form of the wave equations. We then use our formalization for the formal analysis of a terminated transmission line and its special cases, i.e., short- and open-circuited transmission lines commonly used in antenna design [79], by formally verifying the load impedance and the voltage reflection coefficient. In addition, we use our formalization of the telegrapher's equations to formally analyze more complex case studies, such as coupled and cascaded transmission lines, which are widely used in practice.

1.6 Thesis Contributions

The main contribution of this thesis is the development of a methodology and the required formal framework for conducting a theorem proving approach that can handle the analysis of PDEs used in real-world systems within the trusted kernel of a higher-order logic theorem prover. We propose this technique as an alternative approach to less accurate and/or less scalable techniques like computer simulation and paper-and-pencil analysis. To achieve this goal, we formalize several fundamental PDEs including the heat, Laplace, telegrapher's and wave equations. Each library has the formalized mathematical foundations for these PDEs, the formal verification of some of their classical properties, as well as the formal verification of analytical solutions and associated applications. We list below the main contributions of this work with references to related publications provided in the Biography section at the end of the thesis:

- Formalization of the heat conduction problem for a rectangular slab. We formally model the heat transfer as a one-dimensional heat equation for a rectangular slab using the HOL Light theorem prover. Furthermore, the convergence of the general solution is formally verified, along with the correctness of the solution. This formalization can also be useful to develop the formal analysis of a one-dimensional composite slabs [Bio-Cf4].

- Formalization of the Laplace equation and potential flows. The work includes the formal modeling and verification of the validity of the exact potential flow solutions for the Laplace equation in HOL Light. In addition, we formally verify several applications such as the Rankine oval, flow past a circular cylinder and flow past a rotating circular cylinder, each of which involves combining these standard flows to model more complex fluid dynamics [Bio-Cf1].

- Formalization of the telegrapher's and their alternate representations, in the form of wave equations in both time and phasor domains. We present the formal proof for the general solutions of the equations in the phasor domain. The work also includes the formal verification of the relation between the phasor and the time-domain functions in order to formally verify the general solutions of the time-domain PDEs for the current and voltage in an electric transmission line. Moreover, we conduct a formal analysis of a terminated transmission line and its special cases, i.e., short- and open-circuited transmission lines commonly used in antenna design, by formally verifying the load impedance and the voltage reflection coefficient [Bio-Jr1].

- As a more realistic application of the telegrapher's equations, we have investigated coupled transmission lines [Bio-Cf2], capturing the voltages, currents, and their interactions within a system of more complex transmission line configurations. In addition, we have studied the representation of transmission lines using ABCD parameters, based on Kirchhoff's Current Law (KCL) and Kirchhoff's Voltage Law (KVL). This approach was applied to short, medium, and cascaded transmission lines [Bio-Cf3].

1.7 Thesis Organization

The structure of this thesis is as follows: Chapter 2 provides an introduction to the HOL Light theorem prover and its multivariable calculus theories, providing the necessary background for the PDEs analysis of physical systems. This chapter introduces the essential notation and concepts that will be used throughout the thesis.

In Chapter 3, we present the formalization of the heat conduction problem in a rectangular slab, including the one-dimensional heat equation and the formal verification of its linearity. This chapter also contains the formal verification of the solutions of ODEs obtained through the method of separation of variables. Additionally, we provide the formalization of the infinite series solution, along with a formal proof of its convergence. The chapter concludes with the formal verification of the correctness of the general solution of the one-dimensional heat equation.

Chapter 4 presents the formalization of the Laplace equation and potential flows. We present the formalization of various standard potential flows, such as uniform, source/sink, doublet, and vortex flows. Additionally, we present the formal verification of the validity of these exact potential flow solutions for the Laplace equation. To illustrate the effectiveness of our formalization, we provide the formal analysis of several practical applications, including the Rankine oval, potential flow around a circular cylinder, and potential flow past a rotating circular cylinder.

In Chapter 5, we provide the formalization of the telegrapher's equations and their alternative representation as wave equations in both the phasor and time domains. The chapter provides a formal verification of the relation between the telegrapher's and wave equations in the phasor domain, along with the formal verification of their analytical solutions. It also establishes the relationship between phasor and time-domain functions to formally verify the general solutions of time-domain PDEs. We then use our formalization to formally analyze several applications of transmission lines, including terminated, coupled and cascaded transmission lines.

Finally, Chapter 6 concludes the thesis and outlines some future research directions.

Chapter 2

Preliminaries

In this section, we give a brief introduction to the HOL Light theorem prover as well as an overview of some of the fundamental formal definitions and notations of the multivariate calculus theories of HOL Light that are necessary for understanding the rest of the thesis.

2.1 HOL Light Theorem Prover

The HOL Light theorem prover [80] is a mechanized proof-assistant to construct mathematical proofs in higher-order logic [26]. It is implemented in OCaml [81], which is a variant of the Meta-Language functional programming language [82]. HOL Light has a very small logical kernel, which includes some basic axioms and primitive inference rules. Soundness is guaranteed by ensuring that every new theorem is verified by applying these basic axioms and inference rules or any other previously verified theorems/inference rules. In HOL Light, which is based on classical logic, a *theory* comprises types, constants, axioms, definitions, and theorems. HOL Light supports two interactive proof methods: forward and backward. In a forward proof, users begin with theorems that have already been proven and apply inference rules to arrive at the desired theorem. On the other hand, a backward or goal-directed proof method is the opposite of the forward approach. It relies on the concept of *tactics*, which are OCaml functions that reduce the goals into more manageable subgoals, which are verified to conclude with the proofs of theorems. Furthermore, HOL Light contains lemmas, which are proved as part of the more extensive proof process for theorems. The user can choose to either utilize established lemmas or prove new lemmas as they work towards their main objective of proving the theorems. One of the important features of HOL Light is the availability of many automatic proof procedures that help users in conducting proofs in an efficient manner. Table 2.1 summarizes some HOL functions and symbols and their meanings that are used in this thesis. For better readability, we use a mix of HOL Light code and mathematical notation.

Table 2.1: HOL Light Symbols

HOL Light Symbols	Standard Symbols	Description
`@x.t(x)`	εx. t(x)	Some x such that t(x) is true
`λx.t`	$\lambda x.t$	Function that maps x to $t(x)$
`&a`	$\mathbb{N} \rightarrow \mathbb{R}$	Type casting from Natural numbers to Reals
`&num`	$\{0, 1, 2..\}$	Natural numbers data type
`~`	not	Logical *negation*
`--`	-x	Unary *negation*
`Cx(a)`	$\mathbb{R} \rightarrow \mathbb{C}$	Type casting from Reals to Complex
`real`	$\mathbb{R}$	Real data type
`complex`	$\mathbb{C}$	Complex data type
`SUC n`	$(n+1)$	Successor of natural number
`EL n L`	*element*	n^{th} element of list L
`z$i`	z_i	i^{th} component of vector z
`exp x`	e^x	Exponential function (real-valued)
`abs x`	$\|x\|$	Absolute function
`sqrt x`	$\sqrt{x}$	Real square root function
`csqrt x`	$\sqrt{x}$	Complex square root function

2.2 Mathematical Libraries of HOL Light

Harrison [80] formalized a vast body of analysis and topology using the HOL Light proof assistant. We now introduce the two main libraries needed for our formalization of PDEs in HOL Light.

2.2.1 Real Analysis Library

This library includes the formalization of properties of real numbers and a large number of theorems about N-dimensional Euclidean space [63]. Here, we give some fundamental concepts such as real summation, real summability, real differentiation and infinite summation that will be used in the rest of the thesis.

Definition 2.1. *Real Derivative*

```
⊢ ∀f x.  real_derivative f x =
            (@f'.  (f has real derivative f') (atreal x))
```

The function `real_derivative` accepts a real valued function `f` that needs to be differentiated and a real number `x`, and provides the derivative of `f` with respect to `x`. It is formally represented in functional form using the Hilbert choice operator @ [83]. The function `has_real_derivative` expresses the same functionality in relational style.

Definition 2.2. *Higher Real Derivative*

```
⊢ ∀f x.  higher_real_derivative 0 (f:real→real) (x:real) = f x ∧
   (∀n.  higher_real_derivative (SUC n) (f:real→real) (x:real) =
            (real_derivative (λx.  higher_real_derivative n f x) x))
```

The HOL Light function `higher_real_derivative` accepts an order `n` of the derivative, a real-valued function `f` and a real number `x`, and provides a higher-order derivative of order `n` for the function `f` with respect to `x`.

The infinite summation over a function `f`: $\mathbb{N} \rightarrow \mathbb{R}$ is formalized in HOL Light as follows:

Definition 2.3. *Real Sums*

```
⊢ ∀s f L. real_sums (f real_sums l) s ⇔
        ((λn.  sum (s INTER (0..n)) f) ---> l) sequentially
```

The HOL Light function `real_sums` accepts a set of natural numbers `s`: $\mathbb{N} \rightarrow$ `bool`, a function `f`: $\mathbb{N} \rightarrow \mathbb{R}$ and a limit value `l`: $\mathbb{R}$, and returns the traditional mathematical expression $\sum_{k=0}^{\infty} f(k) = l$. Here, the symbol `--->` is used to denote that as n approaches infinity, the sequence of sums converges to l. Moreover, `INTER` captures the intersection of two sets. Similarly, `sequentially` represents a net providing a sequential growth of a function f, i.e., $f(k), f(k+1), f(k+2), ...$, etc. This is mainly used in modeling the concept of an infinite summation.

We provide the formalization of the summability of a function `f`: $\mathbb{N} \rightarrow \mathbb{R}$ over `s`: $\mathbb{N} \rightarrow$ `bool`, which ensures that there exist some limit value `l`: $\mathbb{R}$, such that $\sum_{k=0}^{\infty} f(k) = l$ in HOL Light as:

Definition 2.4. *Real Summability*

```
⊢ ∀s f.  real_summable s f = ∃l.  (f real_sums l) s
```

Now, we provide a formalization of an infinite summation, which will be used in the formal analysis of the heat conduction problem in Chapter 3 of the thesis.

Definition 2.5. *Real Infsum*

```
⊢ ∀s f.  real_infsum s f = @l.  (f real_sums l) s
```

where the HOL Light function `real_infsum` accepts `s`: `num` $\rightarrow$ `bool` specifying the starting point and a function `f` of data-type $\mathbb{N} \rightarrow \mathbb{R}$, and returns a limit value `l`: $\mathbb{R}$ to which the infinite summation of `f` converges from the given `s`.

An infinite summation of a real-valued function Definition 2.5 can be mathematically expressed in an alternate form as follows:

$$\sum_{w=0}^{\infty} f_w(x) = \lim_{N\to\infty} \sum_{w=0}^{N} f_w(x)$$

We proved this equivalence in HOL Light as follows:

Theorem 2.1. *Alternate Representation of an Infinite Summation*

```
⊢ ∀f k s.  real_infsum s (λw.  f w x) =
    reallim sequentially (λk.  sum (s INTER (0..k))(λw.  f w x))
```

2.2.2 Complex Analysis Library

This library encompasses the formalization of fundamental concepts in complex analysis, including complex path integrals and Cauchy's theorem [84]. We now present complex versions of the usual transcendental functions and the key concept of complex differentiability, which we utilize in our formalization.

Definition 2.6. *Re and Im*

$\vdash_{def}$ ∀z. `Re z = z$1`

$\vdash_{def}$ ∀z. `Im z = z$2`

The functions `Re` and `Im` represent the real and imaginary parts of a complex number, respectively. Here, the notation `z$i` represents the i^{th} component of a vector `z`.

Definition 2.7. *Cx and ii*

$\vdash_{def}$ ∀a. `Cx a = complex (a, &0)`

$\vdash_{def}$ `ii = complex (&0, &1)`

`Cx` is a type casting function with a data-type `Cx`: $\mathbb{R} \to \mathbb{C}$. It accepts a real number and returns its corresponding complex number with the imaginary part as zero. Also, the types $\mathbb{R}^2$ and $\mathbb{C}$ are synonymous. The `&` operator has data-type $\mathbb{N} \to \mathbb{R}$ and is used to map a natural number to a real number. Similarly, the function `ii` (iota) represents a complex number with a real part equal to 0 and the magnitude of the imaginary part equal to 1.

Definition 2.8. *Exponential Functions*

$\vdash_{def}$ `∀x. exp x = Re (cexp (Cx x))`

The HOL Light functions `exp` and `cexp` with data-types $\mathbb{R} \to \mathbb{R}$ and $\mathbb{C} \to \mathbb{C}$ represent the real and complex exponential functions, respectively.

Definition 2.9. *Complex Derivative*

```
⊢def ∀f x. complex_derivative f x =
            (@f'. (f has_complex_derivative f') (at x))
```

The function `complex_derivative` describes the complex derivative in functional form. It accepts a function `f`: $\mathbb{C} \to \mathbb{C}$ and a complex number `x`, which is the point at which `f` has to be differentiated, and returns a variable of data-type $\mathbb{C}$, providing the derivative of `f` at `x`. Here, the term `at` indicates a specific point at which the differentiation is being evaluated, namely, at the value of `x`.

Definition 2.10. *Higher Complex Derivative*

```
⊢def ∀f x.
higher_complex_derivative 0 f x = f x ∧
(∀n. higher_complex_derivative (SUC n) f x
     = (complex_derivative (λx. higher_complex_derivative n f x) x))
```

The function `higher_complex_derivative` represents the n^{th}-order derivative of the function `f`. It accepts an order `n` of the derivative, a function `f`: $\mathbb{C} \to \mathbb{C}$ and a complex number `x`, and provides the n^{th} derivative of `f` at `x`.

Chapter 3

Formalization of the Heat Equation

This chapter presents the higher-order logic formalization of a one-dimensional heat equation and the formal verification of the linearity of the heat operator. Additionally, we conduct a formal analysis of the one-dimensional heat conduction problem in a rectangular slab by formally verifying the analytical solution of the heat equation. Furthermore, the chapter addresses the formal verification of the convergence of the generalized solution.

3.1 Heat Equation

The heat equation [8] is one of the most important parabolic PDEs with numerous applications. The phenomenon of heat transfer/propagation can occur by three different means, namely, heat conduction [8], convection [85], and thermal radiation [86]. Heat conduction or diffusion is the flow of energy in a system/body from the region of high temperature to the region of low temperature by direct collision of molecules. Convection, on the other hand, refers to the transfer of the energy due to the physical movement of a bulk fluid. Thermal radiation is the transfer of energy in the form of electromagnetic wave. Heat conduction is the most important type of heat transfer and it is commonly used to analyze problems arising in the design and operation of industrial appliances, such as heat exchanger and compressors. In this context, the heat equation is one of the most well-known PDEs that captures the temperature distribution and diffusion of heat within a body.

The heat equation can be mathematically expressed as follows:

$$\frac{\partial u}{\partial t} = c\frac{\partial^2 u}{\partial x^2} \tag{3.1}$$

This can be derived via conservation of energy and Fourier's law of heat conduction [87]. The constant c is the material's thermal diffusivity:

$$c = \frac{\kappa_0}{s\rho}, \tag{3.2}$$

where κ_0 = thermal conductivity, s = specific heat and ρ = density. This equation is also known as the diffusion equation. To completely determine the function u, which represents the temperature distribution within the medium, it is essential to specify initial and boundary conditions. Subsequently, applying suitable techniques is necessary to obtain analytical solutions to this partial differential equation.

In the following section, we provide a brief review of the heat conduction problem.

3.2 Brief Review of Heat Conduction

Heat conduction is a phenomenon of energy transfer that occurs due to differences in temperature in adjacent components of a body/system. This energy disperses from regions of higher temperature to regions of lower temperature until the system reaches thermal equilibrium. The heat conduction or temperature variation can be mathematically defined as a function of space and time. Generally, the heat conduction in a body is three dimensional i.e., the conduction is significant in all three dimensions and a temperature variation in a body can be modeled as $T = T(x, y, z, t)$. The heat conduction is said to be two-dimensional when the conduction is significant in two-dimensions and negligible in the third dimension. Similarly, it is one-dimensional when the conduction is significant in one-dimensional only and the temperature variable can be modeled as $T = T(x, t)$. The first step for analyzing the heat conduction in a given system/body is to construct a mathematical model of the dynamics of the system, such as heat distribution using the heat equation, which is a PDE. These dynamics provide the variation of the temperature as a function of position/space and time within the heat conducting system/body. The heat distribution (temperature field) usually depends on boundary conditions, initial conditions, material properties, and the geometry of the body. The next step in the heat conduction analysis is to find the solution of the heat equation modeled in the first step that can be obtained by determining a temperature distribution that is consistent with the initial and boundary conditions.

In the next section, we present our methodology for the formalization of the heat equation in higher-order logic.

3.3 Proposed Methodology

The proposed framework, illustrated in Figure 3.1, presents the approach for formally analyzing the heat conduction problem that is modeled by the heat equation. The inputs to this framework, represented by a rectangle and a rectangle with a curved bottom, include the mathematical representation of the heat equation that models the heat conduction problem, along with the initial and boundary conditions necessary for solving this PDE. As depicted in Figure 3.1, the first step is to formalize the heat equation, which captures the heat conduction within the system or body, using the multivariate calculus theories of the HOL Light theorem prover.

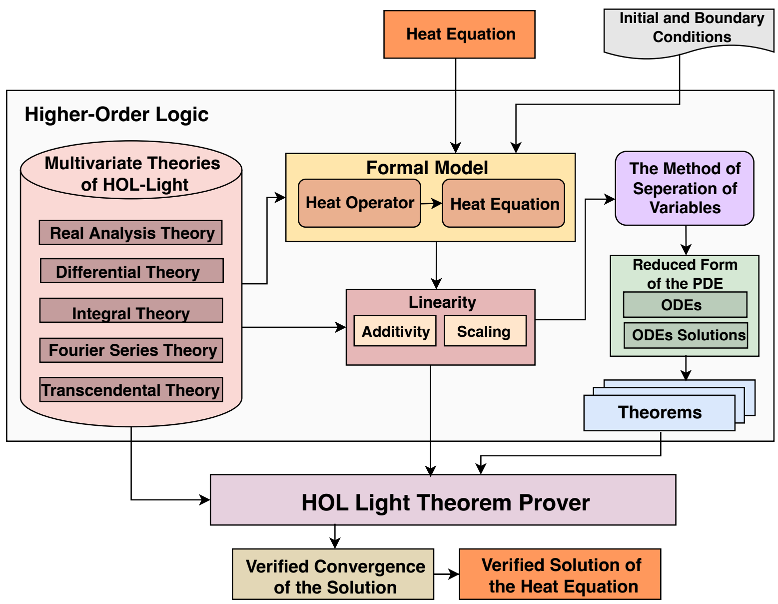


Figure 3.1: Proposed Methodology for the Heat Equation

This also requires the formalization of the heat operator, which is the mathematical operator used in the heat equation. The next step is to formally verify the linearity of the heat equation by proving its additivity and scaling properties. Subsequently, we apply the method of separation of variables to formally model the obtaining ordinary differential equations (ODEs), which are the reduced form of the PDE and their solutions, as shown in Figure 3.1. The next step is to verify a set of theorems establishing the solutions for the ODEs. This is followed by the formalization of the generalized solution, which is a linear combination of the non-trivial solutions. Thereafter, we conduct a formal proof for the convergence of the generalized solution. The final step involves the formal verification of the generalized solution. Finally, the output of the theorem proving based framework of the heat equation provides the verified solution of the PDE model and its convergence.

3.4 Formalization of the Heat Equation

The heat conduction problem for a rectangular slab having a thickness L is depicted in Figure 3.2 [88], which is considered it as a one-dimensional heat conduction problem. Here, the function $u(x,t)$ provides the temperature in the slab at a point x and time t [89].

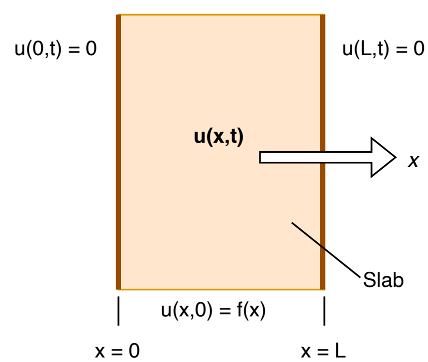


Figure 3.2: Heat Conduction Across the Thickness of a Slab

We can mathematically express the one-dimensional heat conduction (temperature variation) in the rectangular slab as follows [88]:

$$\frac{\partial u(x,t)}{\partial t} = c\frac{\partial^2 u(x,t)}{\partial x^2} \qquad 0 < x < L, \quad t > 0 \tag{3.3}$$

where c is the thermal diffusivity of the slab that depends on the material used for constructing the slab. Equation (3.3) can be equivalently written as:

$$\frac{\partial u(x,t)}{\partial t} - c\frac{\partial^2 u(x,t)}{\partial x^2} = 0$$

Moreover, the solution of the heat equation (Equation (3.3)) should satisfy the following initial and boundary conditions.

Initial Condition:

$$u(x,t) \mid_{t=0} = u(x,0) = f(x) \tag{3.4}$$

Boundary Conditions:

$$u(x,t) \mid_{x=0} = u(0,t) = 0 \tag{3.5}$$

$$u(x,t) \mid_{x=L} = u(L,t) = 0 \tag{3.6}$$

The heat equation (Equation (3.3)) along with Equations (3.4), (3.5) and (3.6) is known as the initial boundary-value problem [90]. It becomes an initial-value problem with respect to time that considers the only initial condition represented by Equation (3.4). Whereas, in the case of its dependence on space only, it represents a boundary-value problem by incorporating the two boundary conditions expressed Equations (3.5) and (3.6).

The heat equation (Equation (3.3)) capturing the one-dimensional heat conduction in a rectangular slab can be formalized in HOL Light as follows:

Definition 3.1. *The Heat Equation*

```
⊢def heat_equation u(x,t) c ⇔ heat_operator u(x,t) c = &0
```

where heat_equation accepts a function u of type ($\mathbb{R} \times \mathbb{R} \to \mathbb{R}$), a space variable x:$\mathbb{R}$, a time variable t:$\mathbb{R}$ and the thermal diffusivity constant c, and returns the corresponding heat equation. The function heat_operator is formalized as follows:

Definition 3.2. *Heat Operator*

```
⊢def ∀u x t.  heat_operator u(x,t) c =
              higher_real_derivative 1 (λt. u(x,t)) t -
                 c * higher_real_derivative 2 (λx. u(x,t)) x
```

Linearity of the Heat Operator

The heat equation is a second-order, homogeneous, linear, parabolic partial differential equation. The linearity of the heat equation is a key property, enabling the use of the principle of superposition [3]. According to this principle, if u and v are solutions to a linear homogeneous equation, then any linear combination, $au + bv$, also satisfies the linear homogeneous equation [3].

The linearity property can be mathematically expressed as follows:

Linearity: An operator L is linear if and only if

$$L[au + bv] = aL[u] + bL[v] \tag{3.7}$$

for any two functions u and v and constants a and b.

We verify this property as the following HOL Light theorem:

Theorem 3.1. *Linearity of the Heat Operator*

```
⊢thm ∀u x a b.
 [A1] (∀t.  (λt. u(x,t)) real_differentiable atreal t) ∧
 [A2] (∀t.  (λt. v(x,t)) real_differentiable atreal t) ∧
```

```
 [A3] (∀x.  (λx.  u(x,t)) real_differentiable atreal x) ∧
 [A4] (∀x.  (λx.  v(x,t)) real_differentiable atreal x) ∧
 [A5] (∀x.  (λx.  real_derivative (λx. u(x,t)) x)
                           real_differentiable atreal x) ∧
 [A6] (∀x.  (λx.  real_derivative (λx.  v(x,t)) x)
                            real_differentiable atreal x)

⇒ (heat_operator (λ(x,t). a * u(x,t) + b * v(x,t)) (x,t) c =

     a * heat_operator (λ(x,t).  u(x,t)) (x,t) c +

       b * heat_operator (λ(x,t).  v(x,t)) (x,t) c)
```

Assumptions `A1` and `A2` ensure that the real-valued functions `u` and `v` are differentiable at `t`, respectively. Assumptions `A3` and `A4` assert the differentiability of the functions `u` and `v` at `x`, respectively. Similarly, Assumptions `A5` and `A6` provide the differentiability conditions for the derivatives of the functions `u` and `v` at `x`, respectively. The proof of the above theorem is mainly based on the properties of derivative and differentiability of real-valued functions.

3.5 Formal Verification of the Solution of the Heat Equation

When the partial differential equation and the boundary conditions are linear and homogeneous, the method of separation of variables can be applied. The idea behind this method is to rewrite the partial differential equation so that the variables are expressed on separate sides of the equation. Using this approach, partial differential equations can be transformed into a set of ODEs. In the following, we formally verify the solutions of ODEs obtained through this technique in HOL.

3.5.1 Verification of the Solutions of ODEs

To find out the solution of the boundary-value problem, i.e., heat equation alongside the boundary conditions Equations (3.3), (3.5) and (3.6), we use the method of separation of variables. By this method, we can mathematically express the solution of the heat equation $u(x,t)$ as a separable equation as follows:

$$u(x,t) = X(x)W(t) \tag{3.8}$$

where X and W are functions of x and t, respectively. We formalize Equation (3.8) in HOL Light as follows:

Definition 3.3. *Separable*

$\vdash_{def}$ ∀u X W t x. separable u x t X W = X(x) * W(t)

By using Equation (3.8) in the heat equation (Equation (3.3)) and after simplification, we obtain the following equation:

$$\frac{1}{c}\frac{\partial[X(x)W(t)]}{\partial t} = \frac{\partial^2[X(x)W(t)]}{\partial x^2} \tag{3.9}$$

Next, using the property of the partial derivative of a separable function transforms the above equation as follows:

$$\frac{1}{c}\frac{dW(t)}{dt}X(x) = W(t)\frac{d^2X(x)}{dx^2} \tag{3.10}$$

We formally verify the equivalence of the left-hand-sides of Equations (3.9) and (3.10) as the following HOL Light theorem.

Theorem 3.2. *Equivalence of Partial and Simple Derivatives (Left-hand Side)*

```
⊢thm ∀u X x W t.
 [A1] (X real_differentiable atreal t) ∧
 [A2] (W real_differentiable atreal t)
  ⇒ (real_derivative (λt.  separable u x t X W) t) =
                                   real_derivative W t * X x
```

Assumptions A1 and A2 provide the differentiability of the functions X and W at t, respectively. The proof process of the above theorem is mainly based on the properties of derivatives and differentiability of the real-valued functions alongwith some arithmetic reasoning. Similarly, we formally verify the equivalence of the right-hand-sides of Equations (3.9) and (3.10) as follows:

Theorem 3.3. *Equivalence of Partial and Simple Derivatives (Right-hand Side)*

```
⊢thm ∀u X x W t.
 [A1] (∀x.  X real_differentiable atreal x) ∧
 [A2] (∀x.  W real_differentiable atreal x) ∧
 [A3] (λx.  real_derivative X x) real_differentiable atreal x
  ⇒ higher_real_derivative 2 (λx.  (separable u x t X W)) x =
                     W t * higher_real_derivative 2 (λx.  X x) x
```

Assumptions A1 and A2 are very similar to those of Theorem 3.2. Assumption A3 ensures that the first-order derivative of the real-valued function X is differentiable at x. The verification of Theorem 3.3 is similar to that of Theorem 3.2.

Now, after rearranging various terms, Equation (3.10) can be expressed as follows:

$$\frac{1}{c}\frac{dW(t)}{dt}\frac{1}{W(t)} = \frac{1}{X(x)}\frac{d^2X(x)}{dx^2} = -\beta^2 \tag{3.11}$$

where the left- and right-hand sides are functions of only t and x, respectively. The equivalence of these two functions of different variables is only possible when both are equal to some constant, which is represented by $-\beta^2$ in the above equation.

Equation (3.11) can be equivalently represented by the following two ordinary differential equations.

$$\frac{d^2X(x)}{dx^2} + \beta^2 X(x) = 0 \tag{3.12}$$

and

$$\frac{dW(t)}{dt} + c.\beta^2 W(t) = 0 \tag{3.13}$$

Now, our problem of solving a boundary-value problem given by Equations (3.3), (3.5), and (3.6) has been transformed into solving a set of linear homogeneous differential equations with constant coefficients, as described by Equations (3.12) and (3.13). Moreover, the solution of the heat equation (Equation (3.3)) can be obtained by multiplying the solutions of these two equations.

The solution of Equation (3.12) is mathematically expressed as:

$$X(x) = Acos(\beta x) + Bsin(\beta x) \tag{3.14}$$

where A and B are the arbitrary constants that can be computed by applying the boundary conditions. Similarly, the solution of the second differential equation (Equation (3.13)) is mathematically described as:

$$W(t) = Ce^{-\beta^2 ct} \tag{3.15}$$

where C is the constant of integration and can be computed by applying the boundary conditions.

We formalize the two differential equations (Equations (3.12) and (3.13)) in HOL Light as follows:

Definition 3.4. *Equation (3.12)*

$\vdash_{def}$ ∀X x b. first_equation X x b ⇔

higher_real_derivative 2 (λx. X (x)) x + b^2 * (λx. X(x)) x = 0

Definition 3.5. *Equation (3.13)*

$\vdash_{def}$ ∀W t b c. second_equation W t b c ⇔

real_derivative (λ*t*. W (t)) t + c * b^2 * W(t) = 0

Similarly, we formalize the solutions of these differential equations in HOL Light as:

Definition 3.6. *Solution of Equation (3.12)*

$\vdash_{def}$ ∀A B x b. first_equation_sol A B x b = A * cos(b * x) + B * sin(b * x)

Definition 3.7. *Solution of Equation (3.13)*

$\vdash_{def}$ ∀C c b t. second_equation_sol C c b t = C * exp (--c * b^2 * t)

Next, we formally verify the solution of the first differential equation (Equation (3.12)) as the following HOL Light theorem:

Theorem 3.4. *Verification of the Solution of Equation (3.12)*

$\vdash_{thm}$ ∀A B x b. (first_equation (λ*x*. first_equation_sol A B x b)) x b

The proof process of the above theorem is based on Definitions 3.4 and 3.5 and properties of real derivative alongside some real arithmetic reasoning.

Similarly, we formally verify the solution of the second differential equation (Equation (3.13)) as follows:

Theorem 3.5. *Verification of the Solution of Equation (3.13)*

$\vdash_{thm}$ ∀C c b t. (second_equation (λt. second_equation_sol C c b t))(t) b c

The proof process of the above theorem is based on Definitions 3.4 and 3.5 and properties of real derivative alongside some real arithmetic reasoning.

To find out the values of the arbitrary constants A and B of the solution of the ordinary differential equation expressed as Equation (3.14), we apply the corresponding boundary conditions. Applying the first boundary condition (Equation (3.5)) results into $A = 0$. Similarly, the application of the second boundary condition (Equation 3.6) provides $Bsin(\beta L) = 0$. We formally verify values of these arbitrary constants based on the corresponding boundary conditions in HOL Light as follows:

Theorem 3.6. *Verification of the Arbitrary Constant A*

$\vdash_{thm}$ `∀A B x b.  x = &0 ∧ first_equation_sol A B x b = &0 ⇒ A = &0`

Theorem 3.7. *Verification of the Arbitrary Constant B*

```
⊢thm ∀A B x b L.
    x = L ∧ A = &0 ∧ first_equation_sol A B x b = &0
    ⇒ first_equation_sol x b A B = B * sin(b * L)
```

The equation $Bsin(\beta L) = 0$ holds if $B = 0$ or $sin(\beta L) = 0$. In case of $B = 0$ alongside $A = 0$, it results into $X(x) = 0$. This further provides $u(x,t) = 0$ as a solution to the heat equation, which is an uninteresting trivial solution. This means that B is equal to some non-zero value, which implies that $sin(\beta L) = 0$. Since β can have infinitely many values for which $sin(\beta L) = 0$ holds, namely $\beta = \beta_w = \frac{\omega\pi}{L}$, this results into a non-trivial solution of the boundary-value problem as follows:

$$u(x,t) = u_w(x,t) = \left[B_w \sin\left(\frac{w\pi x}{L}\right)\right] e^{-\left(\frac{w\pi}{L}\right)^2 ct} \tag{3.16}$$

Now, assume that the function $f(x)$ in the initial condition Equation (3.4) is a linear combination of the function $sin(\frac{w\pi x}{L})$, i.e., Fourier sine series representation as follows:

$$f(x) = \sum_{w=1}^{\infty} B_w \sin\left(\frac{w\pi x}{L}\right) \tag{3.17}$$

3.5.2 Formalization of the Generalized Solution

We can mathematically express the general solution of the heat equation as the following equation since it is a linear combination of the non-trivial solutions of the boundary-value problem that satisfies the initial condition expressed in Equation (3.17).

$$u(x,t) = \sum_{w=1}^{\infty} u_w(x,t) = \sum_{w=1}^{\infty} B_w \sin\left(\frac{w\pi x}{L}\right) e^{-\left(\frac{w\pi}{L}\right)^2 ct} \tag{3.18}$$

The constant B_w of the Fourier sine series representation of $f(x)$ can be determined using the orthogonality property of the sine function and is mathematically expressed as follows:

$$B_w = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{w\pi x}{L}\right) dx \qquad w = 1,2,3... \tag{3.19}$$

We first formalize the Fourier sine coefficient in HOL Light as follows:

Definition 3.8. *Fourier Sine Coefficient*

$\vdash_{def}$ ∀f w L. fourier_sine_coefficient f w L =

$\frac{2}{L}$ * (real_integral (real_interval [0,L])(λx. (f x) *

sin($\frac{w * pi * x}{L}$)))

where `fourier_sine_coefficient` accepts a function `f` $: [0, L] \to \mathbb{R}$, a number `w` and the width of the slab `L`, and returns a real number representing the Fourier sine coefficient of the function `f`.

Now, the solution of the heat equation capturing the heat conduction in a rectangular slab can be alternatively expressed as:

$$u(x,t) = \sum_{w=1}^{\infty} u_w(x,t) = \sum_{w=1}^{\infty} \left(\frac{2}{L}\int_0^L f(x)\sin\left(\frac{w\pi x}{L}\right) dx\right) \sin\left(\frac{w\pi x}{L}\right) e^{-\left(\frac{w\pi}{L}\right)^2 ct} \tag{3.20}$$

Next, we formalize the generalized solution of the heat equation (Equation (3.20)) in HOL Light as follows:

Definition 3.9. *Generalized Solution of the Heat Equation*

$\vdash_{def}$ ∀f x t c L. heat_solution f x t c L = real_infsum (from 1)

(λw. (fourier_sine_coefficient f w L) *

$\texttt{exp(--c * (}\frac{\texttt{w * pi}}{\texttt{L}}\texttt{)}^2\texttt{ * t) * sin(}\frac{\texttt{w * pi * x}}{\texttt{L}}\texttt{))}$

3.5.3 Convergence of the Generalized Solution

The convergence of the generalized solution of the heat equation depends on the convergence of the infinite series $u_w(x, t)$ and is mathematically expressed as the following bound on $u_w(x, t)$.

$$|u_w(x,t)| \leq M_w \tag{3.21}$$

where

$$M_w = \left(\frac{2}{L}\int_0^L |f(x)|dx\right) e^{-\left(\frac{w\pi}{L}\right)^2 ct} \tag{3.22}$$

We compute the upper bound M_w using the upper bound on the Fourier coefficient B_w, and the fact that $\left|sin\left(\frac{w\pi x}{L}\right)\right| \leq 1$, along with the following property of the integral:

$$\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx. \tag{3.23}$$

Next, we formally verify the convergence of the generalized solution of the heat equation as the following HOL Light theorem.

Theorem 3.8. *Convergence of the Generalized Solution*

$\vdash_{thm}$ ∀f x c L t.

[A1] &0 < L ∧ [A2] &0 < t ∧ [A3] &0 < c ∧

```
[A4] f absolutely_real_integrable_on real_interval [&0, L]
 ⇒ ((λw.  fourier_sine_coefficient f w L *
        exp(--c * (w * pi / L)^2 * t) sin(w * pi * x / L)))
                real_sums heat_solution f x t c L) (from 1)
```

Assumptions A1-A3 ensure that the width L, the time t and the constant c are positive real values. Assumption A4 provides the absolute integrability of the function f over the interval [0,L]. The conclusion presents the convergence of the generalized solution of the heat equation. The verification of Theorem 3.8 is mainly based on the following two important lemmas about the summability of the bound M_w and the generalized solution alongside some real arithmetic reasoning.

Lemma 3.1. *Summability of the Bound* M_w

```
⊢lem ∀f c L t.
 [A1] &0 < L ∧ [A2] &0 < t ∧ [A3] &0 < c
 ⇒ real_summable (from 1) (λw.  2/L * real_integral (real_interval [&0,L])
                    (λx.  |f(x)|)) * exp(--c * (w * pi / L)^2 * t))
```

Assumptions A1-A3 are the same as those of Theorem 3.8. The conclusion of the above lemma provides the summability of the upper bound M_w. The verification of Lemma 3.1 is mainly based on the Ratio test, a criterion used to evaluate the convergence or divergence of an infinite series [91] along with some real arithmetic reasoning.

Lemma 3.2. *Summability of the Generalized Solution*

```
⊢lem ∀f x c L t.
 [A1] &0 < L ∧ [A2] &0 < t ∧ [A3] &0 < c ∧
 [A4] f absolutely_real_integrable_on real_interval [&0, L]
 ⇒ real_summable (from 1)(λw.  fourier_sine_coefficient f w L *
                    exp(--c * (w * pi / L)^2 * t) * sin(w * pi * x / L)))
```

Assumptions A1-A4 are the same as those of Theorem 3.8. The verification of Lemma 3.2 is mainly based on the Comparison test, which is used to determine the convergence or divergence of an infinite series by comparing it to another series which behavior is already known [91] and Lemma 3.1 along with some real arithmetic reasoning. More details about the verification of these lemmas and the convergence of the generalized solution of the heat equation can be found in the related HOL Light proof script [92].

3.5.4 Verification of the Generalized Solution

In this section, we formally verify some interesting properties involving the derivatives of the general solution with respect to position x and time t that capture the heat conduction (variation of temperature) in the rectangular slab with respect to position and time.

Theorem 3.9. *Derivative of the Generalized Solution with Respect to Time*

$\vdash_{thm}$ ∀f x c L u u'.

[A1] (∀t. ((λw. (fourier_sine_coefficient f w L) * exp(--c * $(\frac{w * pi}{L})^2$ * t) * sin($\frac{w * pi * x}{L}$)) real_sums u(x,t)) (from 1)) ∧

[A2] (∀t. ((λw. --c * $(\frac{w * pi}{L})^2$ * (fourier_sine_coefficient f w L) * exp(--c * $(\frac{w * pi}{L})^2$ * t) * sin($\frac{w * pi * x}{L}$)) real_sums u'(x,t)) (from 1)) ∧

[A3] ((λt. u(x,t)) has_real_derivative u'(x,t)) (atreal t)

⇒ real_derivative (λt. heat_solution f x t c L) t =

real_infsum (from 1) (λw. --c * $(\frac{w * pi}{L})^2$ * (fourier_sine_coefficient f w L) * exp(--c * $(\frac{w * pi}{L})^2$ * t) * sin($\frac{w * pi * x}{L}$))

Assumption A1 provides the condition that the infinite series converges to the function $u(x,t)$. Similarly, Assumption A2 asserts that the derivative of the infinite series with respect to t converges to the derivative of function $u(x,t)$, i.e. $u'(x,t)$. Assumption A3 ensures the function u has derivative $u'(x,t)$ at point t. The verification of the above theorem is mainly based on swapping the operation of differentiation and infinite summation alongwith properties of the infinite summation and derivatives.

Theorem 3.10. *First Derivative of the Generalized Solution with Respect to Space*

$\vdash_{thm}$ ∀f c L u u'.

[A1] (∀t. ((λw. (fourier_sine_coefficient f w L) *

$$\texttt{exp(--c * (}\frac{\texttt{w * pi}}{\texttt{L}}\texttt{)}^2\texttt{ * t) * sin(}\frac{\texttt{w * pi * x}}{\texttt{L}}\texttt{))}$$

real_sums u(x,t)) (from 1)) ∧

[A2] ((∀x. ((λw. (fourier_sine_coefficient f w L) *

$$\texttt{exp(--c * (}\frac{\texttt{w * pi}}{\texttt{L}}\texttt{)}^2\texttt{ * t) * (}\frac{\texttt{w * pi}}{\texttt{L}}\texttt{) * cos(}\frac{\texttt{w * pi * x}}{\texttt{L}}\texttt{)}$$

real_sums u'(x,t))) (from 1) ∧

[A3] ((λx. u(x,t)) has_real_derivative u'(x,t)) (atreal x)

⇒ real_derivative (λx. heat_solution f x t c L) x =

real_infsum (from 1)(λw. (fourier_sine_coefficient f w L) *

$$\texttt{exp(--c * (}\frac{\texttt{w * pi}}{\texttt{L}}\texttt{)}^2\texttt{ * t) * (}\frac{\texttt{w * pi}}{\texttt{L}}\texttt{) * cos(}\frac{\texttt{w * pi * x}}{\texttt{L}}\texttt{)))}$$

The proof process of Theorem 3.10 is very similar to that of Theorem 3.9.

Theorem 3.11. *Second Derivative of the General Solution with Respect to Space*

⊢ ∀f t c L u u' u''.

[A1] (∀x. ((λw. (fourier_sine_coefficient f w L) * exp(--c * $(\frac{\texttt{w * pi}}{\texttt{L}})^2$ * t) *

sin($\frac{\texttt{w * pi * x}}{\texttt{L}}$) real_sums u(x,t)) (from 1)) ∧

[A2] ((∀x. ((λw. (fourier_sine_coefficient f w L) * exp(--c * $(\frac{\texttt{w * pi}}{\texttt{L}})^2$ * t)

* ($\frac{\texttt{w * pi}}{\texttt{L}}$) * cos($\frac{\texttt{w * pi * x}}{\texttt{L}}$)) real_sums u'(x,t))) (from 1) ∧

[A3] ((λx. u(x,t)) has_real_derivative u'(x,t)) (atreal x) ∧

[A4] $((\forall$x. $((\lambda$w. (fourier_sine_coefficient f w L) * exp(--c * $(\frac{\texttt{w * pi}}{\texttt{L}})^2$ * t)

* $(\frac{\texttt{w * pi}}{\texttt{L}})^2$ * --sin($\frac{\texttt{w * pi * x}}{\texttt{L}}$) real_sums u''(x,t))) (from 1) $\wedge$

[A5] $((\lambda$x. u'(x,t)) has_real_derivative u''(x,t)) (atreal x)

$\Rightarrow$ higher_real_derivative 2 (λx. heat_solution f x t c L) x =

real_infsum (from 1) (λw. (fourier_sine_coefficient f w L) *

exp(--c * $(\frac{\texttt{w * pi}}{\texttt{L}})^2$ * t) * $(\frac{\texttt{w * pi}}{\texttt{L}})^2$ * --sin(($\frac{\texttt{w * pi * x}}{\texttt{L}})^2$))

The verification of the above theorem is mainly based on Theorem 3.10 and properties of derivatives along with some arithmetic reasoning.

3.6 Summary and Discussion

In this chapter, we presented a higher-order logic formalization of the heat equation to formally analyze the heat conduction problem in a rectangular slab. We first formally modeled the heat operator and equation using the multivariate theories of HOL Light. Next, we formally verified the linearity of the heat operator since it is a crucial property to apply the superposition principle, allowing the generalized solution to be expressed as an infinite series. Additionally, we formalized the ODEs and their solutions, which were derived using the method of separation of variables, which reduces the PDEs to a set of ODEs. Furthermore, we formally verified the correctness of these ODE solutions within HOL Light. Subsequently, we formalized the generalized solution of the heat equation in the form of infinite series, constructed from the ODE solutions by applying the superposition principle. Following this, we formally verified the convergence of the generalized solution using ratio and comparison tests available in the mathematical libraries of HOL Light. Finally, we conducted a formal proof of the derivatives for the verification of the generalized solution.

An important aspect of our proposed formal analysis of the heat conduction problem, as compared to traditional analysis techniques, is that all verified theorems are of generic nature, i.e., all functions and variables involved in these theorems are universally quantified and thus can be specialized based on the requirement of the analysis of a rectangular slab with any width and corresponding boundary and initial conditions. Another advantage of our proposed approach is the inherent soundness of the theorem proving technique. It ensures that all required assumptions are explicitly present along with the theorem, which are often ignored in conventional simulation based analysis and hence their absence may affect the accuracy of the corresponding analysis. One of the major difficulties in the proposed formalization was the swapping of the infinite summation and the differential operator that is used in the verification of the derivatives of the general solution. The mathematical proofs available in the literature for this swap operation were very abstract and we developed our own formal reasoning. Another challenge was the formal proof of the convergence of the general solution. Proving the convergence can be difficult even in traditional paper-and-pencil proofs, and verifying them in HOL necessitated a meticulous process, with every detail rigorously provided to the system. It is important to note that this is the *first* formal work on the formalization of a one-dimensional heat equation and the verification of its infinite series solution. Moreover, our work can be extended to analyze transient temperature distribution and heat flux in multi-layer slabs, which are commonly used in thermal protection systems.

In this chapter, we conducted the formal analysis of the heat conduction problem. While one-dimensional heat conduction provides a foundational understanding of thermal behavior, it does not account for other types of physical behaviors, such as fluid dynamics. In the next chapter, we present the formalization of the Laplace equation in HOL Light, which is used in the analysis of a wide range of physical systems.

Chapter 4

Formalization of the Laplace Equation

In this chapter, we provide the higher-order logic formalization of the Laplace equation both in Cartesian and polar coordinates and the formal verification of the linearity of the Laplace operator. Furthermore, we conduct the formal specification and verification of standard potential flows solutions which satisfy the Laplace equation. Moreover, we formally analyze several practical applications, including the Rankine oval, potential flow past a circular cylinder and a potential flow past a rotating circular cylinder.

4.1 Laplace Equation

The Laplace equation is a second-order, linear-homogeneous, elliptic partial differential equation which describes physical phenomena at equilibrium such as a steady state temperature distribution, electrostatic potential in absence of charges, gravitational potential in absence of mass [93]. It is arguably one of the most important differential equations in all of applied mathematics. The solutions of the Laplace equation are called harmonic functions which are important in many fields of science, notably fluid dynamics, electromagnetism and astronomy. There is no time dependence in any of the problems mentioned above. Since the solution of the Laplace equation does not depend on time t, the initial conditions are not specified. However, in order to solve the Laplace equation, certain boundary conditions on the bounding curve or surface of the region must be satisfied. As with other PDEs, the Laplace equation can be studied in different coordinate systems such as cartesian, polar, cylindrical, and spherical. The two-dimensional Laplace equation can be mathematically expressed as follows:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{4.1}$$

It may also be written in the form

$$\Delta u = 0 \tag{4.2}$$

where

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{4.3}$$

known as the *Laplace operator* or *Laplacian* applied to the function u. Note that in general, the Laplacian for a function $u(x_1, , x_n)$ in $\mathbb{R}^n \to \mathbb{R}$ is defined to be the sum of the second partial derivatives:

$$\Delta u = \sum_{k=1}^{n} \frac{\partial^2 u}{\partial x_k^2} \tag{4.4}$$

The physical meaning of the Laplace equation is that it is satisfied by the potential of any such field in source-free domains $\mathfrak{D}$ of the Euclidean space $R^n (n \geq 2)$.

Since the solutions of the Laplace equation are harmonic functions, they are pivotal for analyzing various fluid phenomena in potential flow theory, which is a fundamental aspect of fluid dynamics. In the following section, we provide an overview of potential flow theory.

4.2 Brief Review of Potential Flow Theory

Potential flow theory is a fundamental concept in fluid dynamics that employs harmonic functions to analyze a variety of fluid phenomena. A potential flow describes the velocity field as the gradient of a scalar function known as the velocity potential. Moreover, it characterizes the flow as irrotational and incompressible and provides valuable insights into fluid dynamics. This idealization is in close approximation to real-world scenarios and is highly useful in practice. For example, in aerodynamics, the potential flow theory has been indispensable for constructing analytical models that scrutinize the behavior of airflow around airfoils, wings, and other aerodynamic surfaces, thus facilitating the prediction of crucial aerodynamic forces such as lifts [94].

The foundation for solving aerodynamic problems is rooted in the equations governing fluid flow. Although the Navier-Stokes (NS) equations [95] govern fluid motion, their nonlinear nature makes them difficult to solve [78]. As a result, the Laplace equation emerges as a valuable alternative, providing an exact representation of incompressible, inviscid and irrotational flows. This equation forms the cornerstone of potential flow theory, where the stream function and velocity potential, both algebraic functions that satisfy Laplace's equation, are utilized to construct flow fields.

A potential flow can be defined as a steady, incompressible and irrotational flow. A condition that is necessary and sufficient to identify a flow as irrotational:

$$\overrightarrow{\nabla} \times \overrightarrow{V} = 0 \tag{4.5}$$

This indicates that the velocity field **V** is a conservative vector field denoted by the gradient of a scalar velocity potential function (ϕ):

$$\overrightarrow{V} = \overrightarrow{\nabla}\phi \tag{4.6}$$

If the velocity potential is known, then the velocity at any point can be determined using

$$u = \frac{\partial \phi}{\partial x}, \quad v = \frac{\partial \phi}{\partial y} \tag{4.7}$$

The irrotationality condition for two-dimensional flows vorticity is given by

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \xi \tag{4.8}$$

Here, $\xi = 0$ since the flow is irrotational.

Similarly, in the case of an incompressible flow, it follows from the continuity equation that

$$\overrightarrow{\nabla}.\overrightarrow{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{4.9}$$

The two-dimensional continuous flow is described by the stream function (for incompressible flow) ψ, which determines the velocity at any point as:

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x} \tag{4.10}$$

Substituting Equations (4.7) and (4.10) into Equations (4.9) and (4.8), respectively, yields the conditions for continuous irrotational flow:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \tag{4.11}$$

which is the Laplace equation in Cartesian coordinates [94]. The Laplace equation for the stream function can also be written in polar coordinates as:

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2} = 0 \tag{4.12}$$

Several techniques are employed to determine both the velocity potential (ϕ) and the stream function (ψ). For instance, common numerical and analytical techniques such as Finite Element Method (FEM) [96] and separation of variables [15], respectively are commonly used to obtain the solution of the Laplace equation with the appropriate boundary conditions. Furthermore, another efficient technique is to find some simple functions that satisfy the Laplace equation and to model the flow around the body of interest, leveraging the linearity of the equation. We focus on this latter method, which is the most widely used procedure for potential flows.

In the next section, we present our methodology for the formalization of the Laplace equation as well as standard potential flows.

4.3 Proposed Methodology

The proposed approach for formally analyzing the Laplace equation and standard potential flows using higher-order logic theorem proving is depicted in Figure 4.1. The inputs to this framework, represented by a rectangle and an elongated rectangle, consist of the mathematical formulation of the Laplace equation and its exact potential flow solutions.

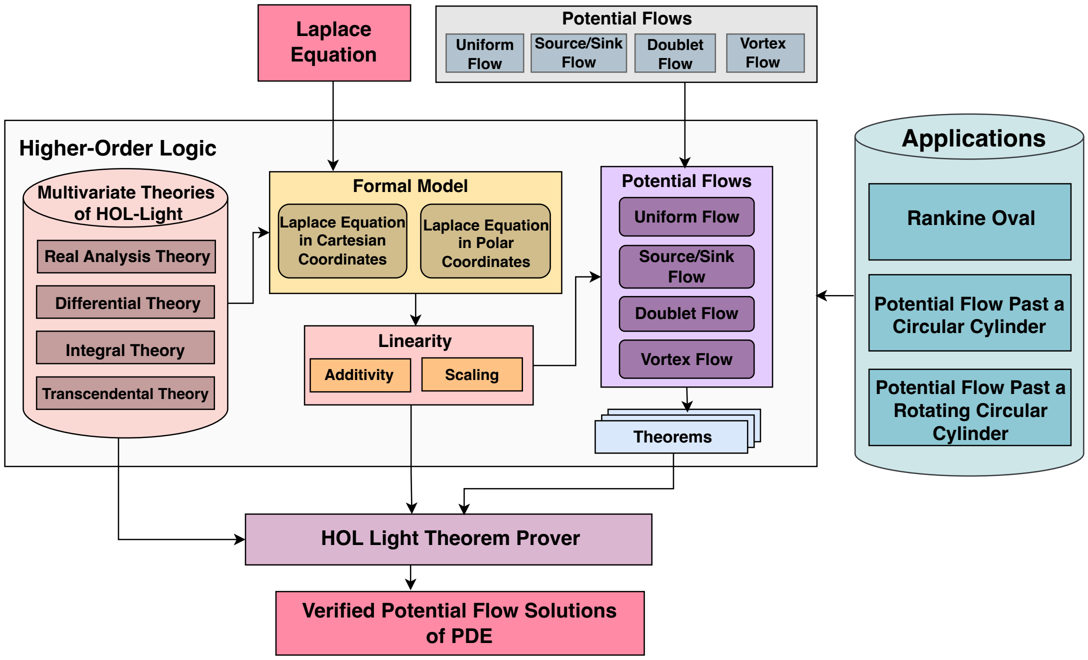


Figure 4.1: Proposed Methodology for the Laplace Equation

The first step in our methodology involves transforming the Laplace equation into its corresponding model in both Cartesian and polar coordinates within higher-order logic, requiring the use of HOL Light's multivariate calculus libraries. The next step is to formally verify the linearity of the Laplace operator, including the verification of its additivity and scaling properties. The linearity property is a critical step for applying the superposition principle which allows us to combine fundamental potential flow solutions for the analysis of more complicated aerodynamic configurations. We then formally model four fundamental potential flows such as the uniform, source/sink, doublet and vortex flows as depicted in Figure 4.1. Subsequently, we establish theorems to facilitate the formal verification of the validity of these exact potential flow solutions which satisfy the Laplace equation. Finally, the output of this framework provides the verified solutions of the Laplace equation. In order to demonstrate the practical applicability of our formalization, we conduct a formal analysis of several practical scenarios, including the Rankine oval, potential flow past a circular cylinder, and potential flow past a rotating circular cylinder, by combining these fundamental flows.

4.4 Formalization of the Laplace Equation

In this section, we present our formalization of the Laplace equation in terms of stream function in both Cartesian and polar coordinates in HOL Light.

Definition 4.1. *Laplace Equation in Cartesian Coordinates*

$\vdash_{def}$ `laplace_equation psi(x,y)` $\Leftrightarrow$ `laplace_operator psi(x,y) = &0`

where `laplace_equation` accepts the real function `psi`: $\mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$, the space variables `x`:$\mathbb{R}$ and `y`:$\mathbb{R}$ and returns the corresponding Laplace equation. It is important to note that the Laplace equation can also be expressed in terms of the velocity potential. Consequently, both the stream function and the velocity potential satisfy the Laplace equation.

The function Laplace operator is formalized as:

Definition 4.2. *Laplace Operator*

```
⊢def ∀psi x y.  laplace_operator psi(x,y) =
                  higher_real_derivative 2 (λx. psi(x,y)) x +
                    higher_real_derivative 2 (λy. psi(x,y)) y
```

Here, `higher_real_derivative` represents the n^{th}-order real derivative of a function.

The formal representation of the Laplace equation in polar coordinates for the stream function, i.e., Equation (4.12) is hence given as follows:

Definition 4.3. *Laplace Equation in Polar Coordinates*

```
⊢def ∀psi r theta.  laplace_in_polar psi r theta =
        higher_real_derivative 2 (λr. psi(r,theta)) r +
         1/r * higher_real_derivative (λr. psi(r,theta)) r +
          1/r^2 * higher_real_derivative (λtheta. psi(r,theta)) theta = &0
```

where the HOL Light function `laplace_in_polar` mainly accepts the function `psi` of type $\mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$, the radial distance `r` and the angle `theta` and returns the corresponding equation. It can similarly be defined for the velocity potential as well.

The next step is to formally verify the linearity of the Laplace operator, an important step for utilizing the superposition principle in the combination of standard potential flows. The linearity of the Laplace operator can be mathematically expressed as follows:

$$\Delta(a\psi + b\phi) = a\Delta\psi + b\Delta\phi \tag{4.13}$$

where Δ represents the Laplace operator introduced in Equation 4.3.

Equation (4.13) is formally verified by the following HOL Light theorem:

Theorem 4.1. *Linearity of Laplace Operator*

```
⊢thm ∀psi phi a b.
 [A1] (∀x.  (λx.  psi(x,y)) real_differentiable atreal x) ∧
 [A2] (∀x.  (λx.  phi(x,y)) real_differentiable atreal x) ∧
 [A3] (∀y.  (λy.  psi(x,y)) real_differentiable atreal y) ∧
 [A4] (∀y.  (λy.  phi(x,y)) real_differentiable atreal y) ∧
 [A5] (∀x.  (λx.  real_derivative (λx.  psi(x,y)) x)
                    real_differentiable atreal x) ∧
 [A6] (∀x.  (λx.  real_derivative (λx.  phi(x,y)) x)
                    real_differentiable atreal x) ∧
 [A7] (∀y.  (λy.  real_derivative (λx.  psi(x,y)) y)
                    real_differentiable atreal y) ∧
 [A8] (∀y.  (λy.  real_derivative (λy.  phi(x,y)) y)
                    real_differentiable atreal y)
```

```
⇒ laplace_operator (λ(x,y).  a * psi(x,y) + b * phi(x,y)) (x,y) =
   a * laplace_operator (λ(x,y).  psi(x,y)) (x,y) +
     b * laplace_operator (λ(x,y).  phi(x,y)) (x,y)
```

Assumptions `A1` and `A2` ensure that the real-valued functions `psi` and `phi` are differentiable at `x`, respectively. Assumptions `A3` and `A4` assert the differentiability of the functions `psi` and `phi` at `y`, respectively. Additionally, Assumptions `A5` and `A6` provide the differentiability conditions for the derivatives of the functions `psi` and `phi` at `x`, respectively. Similarly, Assumptions `A7` and `A8` guarantee the differentiability conditions for the derivatives of the functions `psi` and `phi` at `x`, respectively. The proof of the above theorem relies mainly on the properties of derivatives and the differentiability of real-valued functions.

4.5 Formalization of Potential Flows

In this section, we present some basic functions which satisfy the Laplace equation. Any function that satisfies this equation describes a potential flow. It is important to note that we are interested in employing exact potential flow solutions to formally validate them for the Laplace equation. Moreover, our aim is to employ these elementary flows as foundational components for constructing a desired flow field, rather than deriving the flows themselves.

4.5.1 Uniform Flow

The most basic type of flow is a uniform steady flow as shown in Figure 4.2. A uniform flow directed in the positive x-direction has the velocity components $u = U$ and $v = 0$ everywhere.

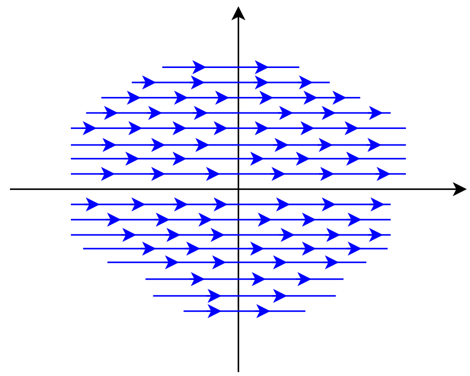


Figure 4.2: Uniform Flow

This type of flow is irrotational and therefore possesses a velocity potential ϕ, which can be shown as follows:

$$\phi = Ux \tag{4.14}$$

Additionally, the stream function can be expressed as:

$$\psi = Uy \tag{4.15}$$

The formal representations of a uniform flow for the stream function and the velocity potential are given as follows:

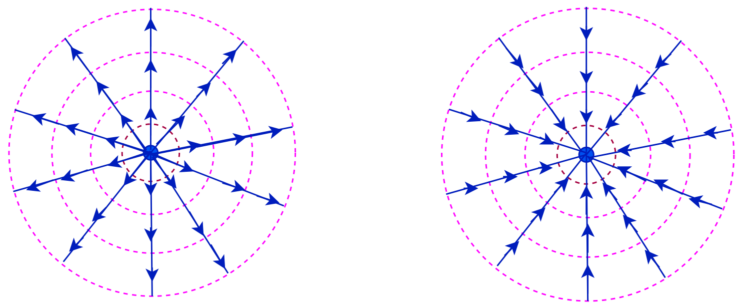
Definition 4.4. *Uniform Flow for the Stream Function and Velocity Potential*

$\vdash_{def}$ `∀U y. stream_uniform U y = U * y`

$\vdash_{def}$ `∀U x. velocity_uniform U x = U * x`

4.5.2 Source/Sink Flow

In two-dimensional fluid dynamics, a source is characterized as a point from which fluid propagates radially outward, whereas a sink is defined as a negative source, exhibiting radial inward fluid motion. These flow patterns are represented in Figures 4.3(a) and 4.3(b), respectively.



(a) Source Flow (b) Sink Flow

Figure 4.3: Source/Sink Flow

The exact potential flow solutions centered at point (x_0, y_0) for the stream function and the velocity potential are mathematically expressed as [94]:

$$\psi(x,y) = \frac{m}{2\pi} \tan^{-1}\left(\frac{y-y_0}{x-x_0}\right) \tag{4.16}$$

$$\phi(x,y) = \frac{m}{4\pi} \ln((x-x_0)^2 + (y-y_0)^2) \tag{4.17}$$

Here, m denotes the strength of the source. A positive m ($m > 0$) denotes a source flow, whereas a negative m ($m < 0$) indicates a sink flow. Note that (x_0, y_0) represents the fixed location of the source or sink. When a parameter is fixed for the entire problem, it is common to treat it as a constant and omit it from the function arguments for simplicity.

Now, we formalize the above equations, i.e., Equations (4.16) and (4.17) in HOL Light as follows:

Definition 4.5. *Source Flow for the Stream Function*

$$\vdash_{def} \forall \texttt{m x y x0 y0.} \quad \texttt{stream\_source m x y x0 y0} = \frac{\texttt{m}}{2 * \texttt{pi}} * \texttt{atn}\left(\frac{\texttt{y} - \texttt{y0}}{\texttt{x} - \texttt{x0}}\right)$$

Definition 4.6. *Source Flow for the Velocity Potential*

$$\vdash_{def} \forall \texttt{m x y x0 y0.} \quad \texttt{velocity\_source m x y x0 y0} = \frac{\texttt{m}}{4 * \texttt{pi}} * \texttt{log}\,((\texttt{x} - \texttt{x0})^2 + (\texttt{y} - \texttt{y0})^2)$$

Here, `atn` and `log` indicate the inverse of the tangent function and the natural logarithm, respectively. It is important to note that, in contrast to the informal definition of the equations [94], HOL Light definitions explicitly list all arguments.

In the next subsections, we will use the polar coordinates r and θ to describe the doublet and vortex flows. Note that uniform and source/sink flows can be similarly represented using polar coordinates, utilizing the relationships $x = r\cos\theta$, $y = r\sin\theta$. These transformations are particularly useful for practical examples.

4.5.3 Doublet Flow

As depicted in Figure 4.4, the doublet is a special flow pattern that arises when a source and a sink of equal strength are constrained to have a constant ratio of strength to distance (κ), as the distance approaches zero. The resulting solutions for the stream function and the velocity potential are as follows:

$$\psi(r,\theta) = -\frac{\kappa}{2\pi r}\sin\theta \tag{4.18}$$

$$\phi(r,\theta) = \frac{\kappa}{2\pi r}\cos\theta \tag{4.19}$$

The next step is to formalize the above equations (Equations (4.18) and (4.19)) in HOL Light:

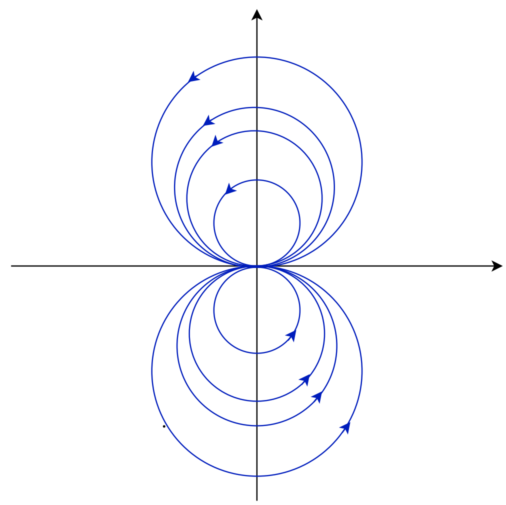


Figure 4.4: Doublet Flow

Definition 4.7. *Doublet Flow for the Stream Function*

$\vdash_{def}$ ∀K theta r. stream_doublet K theta r = $-\dfrac{K}{2 * pi * r}$ * sin(theta)

Definition 4.8. *Doublet Flow for the Velocity Potential*

$\vdash_{def}$ ∀K theta r. velocity_doublet K theta r = $\dfrac{K}{2 * pi * r}$ * cos(theta)

where stream_doublet and velocity_doublet accept the strength K, the radius r and the angle theta and return the corresponding functions.

4.5.4 Vortex Flow

A two-dimensional, steady flow that circulates about a point is known as a line vortex. In this type of flow, the streamlines form concentric circles around a specific point as shown in Figure 4.5. It is important to note that the irrotational nature of the flow is not contradicted by the potential vortex formulation. Fluid elements travel in a circular path around the vortex center without rotating about their axes, thus meeting the condition of irrotational flow.

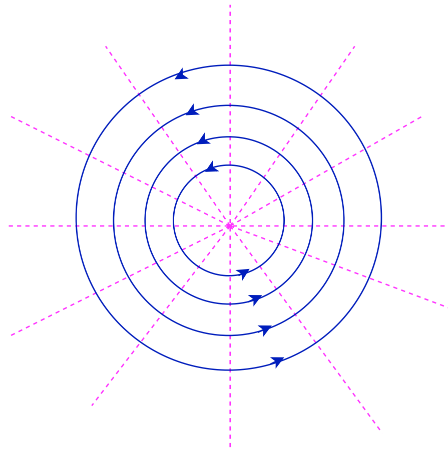


Figure 4.5: Vortex Flow

The exact potential flow solutions centered at the origin are mathematically expressed as:

$$\psi(r, \theta) = \frac{\Gamma}{2\pi} \ln(r) \tag{4.20}$$

$$\phi(r, \theta) = -\frac{\Gamma}{2\pi}\theta \tag{4.21}$$

where Γ represents the circulation, which is often positive when moving counter-clockwise.

Next, we formalize the vortex flow for the stream function and the velocity potential, i.e., Equations (4.20) and (4.21) as:

Definition 4.9. *Vortex Flow for the Stream Function*

$\vdash_{def}$ $\forall$gamma r. stream_vortex gamma r = $\frac{\texttt{gamma}}{2 * \texttt{pi}}$ * log(r)

Definition 4.10. *Vortex Flow for the Velocity Potential*

$\vdash_{def}$ $\forall$gamma theta. velocity_vortex gamma theta = $\frac{-\texttt{gamma}}{2 * \texttt{pi}}$ * theta

4.6 Formal Verification of the Solutions of the Laplace Equation

In this section, we conduct a formal verification of the exact potential flow solutions of the Laplace equation. The purpose of this verification is to ensure the correctness of analytical solutions and then establish their fundamental role in describing fluid behavior and facilitating engineering applications. With the formal definitions outlined previously, an important step is to verify that these potential flow solutions satisfy the Laplace equation. In other words, this is the main condition for potential flows to be valid, which is fundamental for understanding fluid behavior in various contexts. We start with the verification of the source flow for the stream function, i.e., Equation (4.16) in HOL Light as follows:

Theorem 4.2. *Verification of the Source Flow for the Stream Function*

```
⊢thm ∀m x0 y0 psi.
 [A1] (∀x.  x ≠ x0) ∧ [A2] (∀y.  y ≠ y0) ∧
 [A3] (∀x y.  psi(x,y) = stream_source m x y x0 y0)
   ⇒ laplace_equation psi x y
```

Assumptions `A1` and `A2` ensure that the points in a Cartesian coordinate system are different from each other. Assumption `A3` provides the solution of the Laplace's equation for the source flow, i.e., Equation (4.16). The proof of the above theorem is mainly based on the real differentiation of the source flow solution with respect to the parameters `x` and `y`.

We now formally verify the source flow of the velocity potential, i.e., Equation (4.17) satisfy the Laplace equation as follows:

Theorem 4.3. *Verification of the Source Flow for the Velocity Potential*

```
⊢thm ∀m x0 y0 phi.
 [A1] (∀x.  x ≠ x0) ∧ [A2] (∀y.  y ≠ y0) ∧
 [A3] (∀x y.  phi(x,y) = velocity_source m x y x0 y0)
   ⇒ laplace_equation_vel phi x y
```

Assumptions `A1` and `A2` are the same as those in Theorem 4.2. Assumption `A3` provides the source flow for the velocity potential, as described by Equation (4.17). The verification of Theorem 4.3 is similar to that of Theorem 4.2.

Our next step is to formally verify the doublet flow (Equation (4.18)) as the following HOL Light theorem:

Theorem 4.4. *Verification of the Doublet Flow for the Stream Function*

```
⊢thm ∀K psi.
 [A1] (λr.  &0 < r) ∧
 [A2] (∀r theta.  psi(r,theta) = stream_doublet K theta r))
   ⇒ laplace_in_polar psi r theta
```

Assumption `A1` is required to ensure that the radial distance is greater than zero. Assumption `A2` provides the solution of the Laplace equation in polar coordinates (Equation (4.12)) for doublet flow (Equation (4.18)). The verification of Theorem 4.4 is mainly based on the properties of real derivative [97] and some real arithmetic reasoning.

Next, we verify the doublet flow of velocity potential in HOL Light:

Theorem 4.5. *Verification of the Doublet Flow for the Velocity Potential*

```
⊢thm ∀K p hi.
 [A1] (λr.  &0 < r) ∧
 [A2] (∀r theta.  phi(r,theta) = velocity_doublet K theta r))
   ⇒ laplace_vel_polar phi r theta
```

Assumption `A1` is the same as that of Theorem 4.4. Assumption `A2` provides the velocity potential of doublet flow, i.e., Equation (4.19). The verification of Theorem 4.5 is very similar to that of Theorem 4.4.

Finally, we formally verify the vortex flow of the stream function and the velocity potential, as given in Equations (4.20) and (4.21) satisfy the Laplace equation as the following HOL Light theorems:

Theorem 4.6. *Verification of the Vortex Flow for the Stream Function*

```
⊢thm ∀gamma psi.
 [A1] (λr.  &0 < r) ∧
 [A2] (∀r theta.  psi(r,theta) = stream_vortex gamma r))
   ⇒ laplace_in_polar psi r theta
```

Assumption `A1` is the same as that of Theorem 4.5. Assumption A2 presents the vortex flow solution for the stream function, i.e., Equation (4.20) . The conclusion of Theorem 4.6 provides that the vortex flow solution satisfies the Laplace equation. The proof of Theorem 4.6 is primarily based on the real differentiation of the vortex flow solution with respect to the parameters `r` and `theta`.

Theorem 4.7. *Verification of the Vortex Flow for the Velocity Potential*

```
⊢thm ∀gamma phi.
  (∀r theta.  phi(r,theta) = velocity_vortex gamma theta)) ⇒
                                     laplace_vel_polar phi r theta
```

The verification of the above theorem is primarily based on the properties about real derivatives as well as some real arithmetic reasoning.

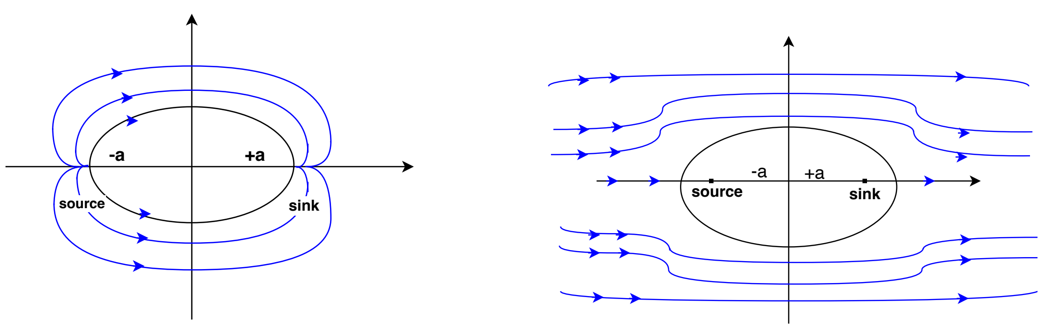
In the next section, we use these formally verified solutions to build more complicated flows which are widely applied in the analysis of flow patterns around an airfoil [98].

4.7 Applications: Formal Analysis of Standard Flows

Thanks to the linearity of the Laplace's equation, more complicated flow fields can be constructed from the superposition of basic solutions. If ψ_1 and ψ_2 are the solutions (stream functions) of the Laplace equation, and their linear combination $\psi_1 + \psi_2$ will also be a solution for a two-dimensional incompressible and irrotational flow. This unique feature makes this equation a powerful tool to analyze fluid flow problems. The ability to obtain new flow patterns by superimposing known flows is fundamental to wing theory, as it provides simple solutions to complex problems [99].

4.7.1 The Rankine Oval

By combining the exact solutions for uniform and source/sink flows, we can construct a flow field around an oval-shaped object. The resultant configuration is known as the Rankine oval which is depicted in Figure 4.6 [78]. We start by analyzing the flow pattern around a source and a sink. The source and sink are placed along the x-axis, separated by a distance of $2a$, as depicted in Figure 4.6(a).



(a) (b)

Figure 4.6: The Rankine Oval

The origin is situated equidistantly between them. We now superimpose the uniform, source and sink flows, all positioned in the x-direction, with a line source located at $(-a, 0)$ and a line sink of equal and opposite strength located at $(+a, 0)$, as depicted in Figure 4.6(b). Assume the strengths of the source and the sink are $+m$ and $-m$, respectively. The overall stream function (ψ) and velocity potential (ϕ) for this combination of flows are expressed as:

$$\psi = \psi_{uniform} + \psi_{source} + \psi_{sink} \tag{4.22}$$

$$\phi = \phi_{uniform} + \phi_{source} + \phi_{sink} \tag{4.23}$$

Mathematically, they are represented by the combination of Equations (4.14), (4.15), (4.16) and (4.17) for the stream function and the velocity potential:

$$\psi(x, y) = -Uy + \frac{m}{2\pi}\left[\arctan\left(\frac{y}{x+a}\right) - \arctan\left(\frac{y}{x-a}\right)\right] \tag{4.24}$$

$$\phi(x, y) = Ux + \frac{m}{4\pi}\ln\left(\frac{(x+a)^2 + y^2}{(x-a)^2 + y^2}\right) \tag{4.25}$$

Next, we formally verify these combined flows for the stream function as the following HOL Light theorem:

Theorem 4.8. *Verification of the Rankine Oval for the Stream Function*

```
⊢thm ∀U m a psi x0 x1 y0 y1.
  [A1] (∀x.  x ≠ a) ∧ [A2] (∀x.  x ≠ --a) ∧ [A3] x0 = --a ∧
  [A4] x1 = a ∧ [A5] y0 = &0 ∧ [A6] y1 = &0 ∧
  [A7] (∀x y.  psi(x,y) = sum (0..2) (λn.  EL n [--stream_uniform U y;
                  stream_source m x y x0 y0; stream_sink m x y x1 y1]))
    ⇒ laplace_equation psi x y
```

Assumptions A1 and A2 guarantee the validity of our expression by specifying that x must be different from a and --a, respectively. Asssumptions A3 and A4 provide the distance from the origin. Assumptions A5 and A6 assert that the points y0 and y1 are equal to zero since the flows are oriented in towards the x-direction. Assumption A7 provides the combined solutions for the stream function, i.e., Equation (4.24). Here, the function EL n l extracts the n^{th} element from a list l. The verification of Theorem 4.8 is mainly based on the properties of real derivatives, some real arithmetic reasoning and the following HOL Light lemma:

Lemma 4.1. *Superposition of the Solutions*

```
⊢lem ∀U m x y x0 x1 y0 y1.
sum (0..2) (λn.  EL n [--stream_uniform U y; stream_source m x y x0 y0;
   stream_sink m x y x1 y1]) = --stream_uniform U y +
                           stream_source m x y x0 y0 + stream_sink m x y x1 y1
```

The above lemma states that the summation of the list equals to the linear combination of uniform, source and sink flows.

Now, we formally verify the combined flows, i.e., Equation (4.25), for the velocity potential in HOL Light as follows:

Theorem 4.9. *Verification of the Rankine Oval for the Velocity Potential*

```
⊢thm ∀U m a phi x0 x1 y y0 y1.
 [A1] (∀x.  a < x) ∧ [A2] (∀x.  --a < x) ∧ [A3] &0 < y ∧
 [A4] x0 = --a ∧ [A5] x1 = a ∧ [A6] y0 = &0 ∧ [A7] y1 = &0 ∧
 [A8] (∀x y.  phi(x,y) = sum (0..2) (λn.  EL n [velocity_uniform U x;
                  velocity_source m x x0 y y0; velocity_sink m x x1 y y1))
   ⇒ laplace_equation_vel phi x y
```

Assumptions A1 and A2 guarantee the validity of our expression by specifying that x must be greater than a and --a, respectively. Assumptions A4-A7 are similar to

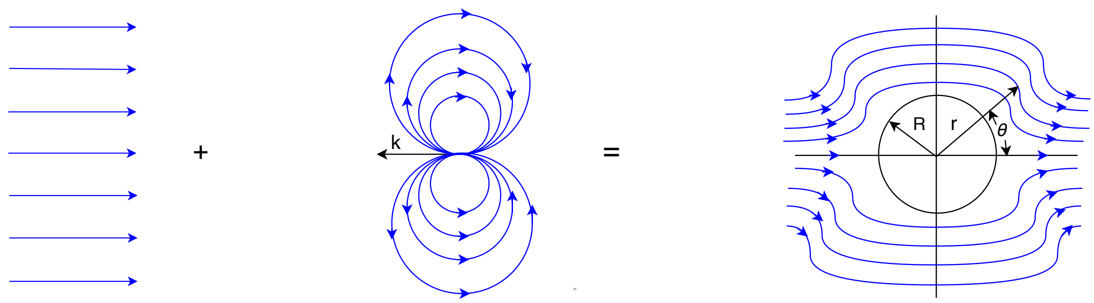
Assumptions A3-A6 stated in Theorem 4.8. The verification of Theorem 4.9 is also very similar to that of Theorem 4.8.

4.7.2 Potential Flow Past a Circular Cylinder

As shown in Figure 4.7 [78], we can build a potential flow solution for the flow around a circular cylinder using the superposition of a uniform (Figure 6(a)) and a doublet flow (Figure 6(b)) in the x-direction. This combination produces a non-lifting flow over the cylinder, as represented in Figure 6(c). The resulting stream function and velocity potential for this particular combination of potential flows can be given as:

$$\psi = \psi_{uniform} + \psi_{doublet} \tag{4.26}$$

$$\phi = \phi_{uniform} + \psi_{doublet} \tag{4.27}$$



(a) Uniform Flow

(b) Doublet Flow

(c) Non-Lifting Flow over a Cylinder

Figure 4.7: Potential Flow Past a Circular Cylinder

We can mathematically express this combination by adding the solutions for uniform and doublet flow, i.e., Equations (4.14), (4.15), (4.18) and (4.19). It is known that $y = rsin\theta$ in polar coordinates.

$$\psi(r,\theta) = U\left(r + \frac{\kappa}{2\pi r}\right)\sin\theta \tag{4.28}$$

$$\phi(r,\theta) = U\left(r - \frac{\kappa}{2\pi r}\right)\cos\theta \tag{4.29}$$

Next, we formally verify Equation (4.28) in HOL Light as follows

Theorem 4.10. *Potential Flow Past a Circular Cylinder for the Stream Function*

```
⊢thm ∀U K y psi.
 [A1] (∀r.  &0 < r) ∧ [A2] (∀r theta.  y = r * sin(theta)) ∧
 [A3] (∀r theta.  psi(r,theta) = sum (0..1) (∀n.  EL n [stream_uniform U y;
                                               stream_doublet K theta r]))
  ⇒ laplace_in_polar psi r theta
```

Assumption A1 ensures that the radial distance is greater than zero, while Assumption A2 indicates that `y = r * sin(theta)` in polar coordinates. Assumption A3 provides the superposition of the uniform and doublet flow solutions for the stream function, i.e., Equation (4.28). Similar to Theorem 4.8, we proved a lemma regarding the superposition of the solutions as well as proving the real derivatives of the solution in order to formally verify this theorem.

Now, we formally verify Equation (4.29) as the following HOL Light theorem:

Theorem 4.11. *Potential Flow Past a Circular Cylinder for the Velocity Potential*

```
⊢thm ∀U K x u.
 [A1] (∀r.  &0 < r) ∧ [A2] (∀r theta.  x = r * cos(theta)) ∧
 [A3] (∀r theta. phi(r,theta) = sum (0..1) (∀n.  EL n [velocity_uniform U x;
                                                 velocity_doublet K theta r]))
  ⇒ laplace_vel_polar phi r theta
```

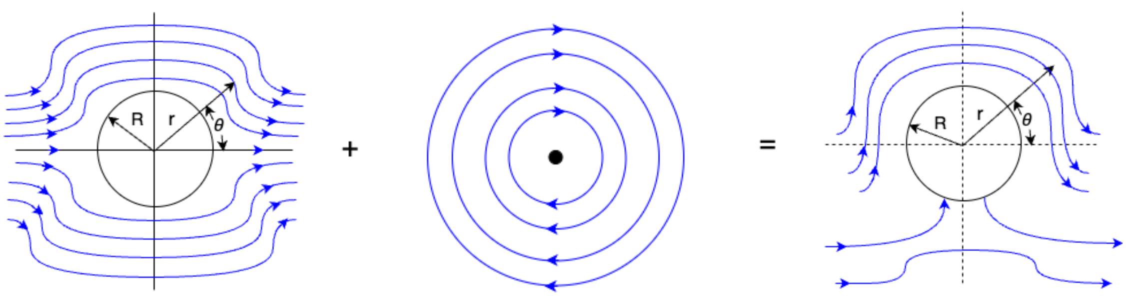
The verification of the above theorem is similar to that of Theorem 4.10.

4.7.3 Potential Flow Past a Rotating Circular Cylinder

Figure 4.8(c) [78] illustrates a flow around a rotating circular cylinder. This flow can be constructed by combining a uniform flow and a doublet flow, as depicted in Figure 4.8(a), along with a vortex flow, as shown in Figure 4.8(b). In this context, the stream function and the velocity potential for this combination of potential flows can, respectively, be given as:

$$\psi = \psi_{uniform} + \psi_{doublet} + \psi_{vortex} \tag{4.30}$$

$$\phi = \phi_{uniform} + \phi_{doublet} + \phi_{vortex} \tag{4.31}$$



(a) Non-Lifting Flow over a Cylinder (b) Vortex Flow (c) Lifting Flow over a Cylinder

Figure 4.8: Potential Flow Past a Rotating Circular Cylinder

It is important to note that combining a uniform flow and a doublet flow effectively models the flow around a non-rotating circular cylinder, as given by Equations (4.28) and (4.29). Therefore, we can write the final mathematical expression of these flows for the stream function and the velocity potential by adding the solutions, i.e., Equations (4.20), (4.21), (4.28) and (4.29) as:

$$\psi(r, \theta) = U\left(r + \frac{\kappa}{2\pi r}\right) sin\theta + \frac{\Gamma}{2\pi}\ln(r) \tag{4.32}$$

$$\phi(r,\theta) = U\left(r - \frac{\kappa}{2\pi r}\right) cos\theta + -\frac{\Gamma}{2\pi}\theta \tag{4.33}$$

The above equations can be alternatively written as:

$$\psi(r,\theta) = Ursin\theta\left(1 - \frac{R^2}{r^2}\right) + \frac{\Gamma}{2\pi}\ln(r) \tag{4.34}$$

$$\phi(r,\theta) = Urcos\theta\left(1 - \frac{R^2}{r^2}\right) + \frac{\Gamma}{2\pi}\theta \tag{4.35}$$

where $R^2 = \dfrac{m}{2\pi U}$ and m is the strength of the doublet.

Now, we formally verify Equation (4.32) as the following HOL Light theorem:

Theorem 4.12. *Flow Past a Rotating Circular Cylinder for the Stream Function*

```
⊢thm ∀U K y gamma psi.
  [A1] (∀r.  &0 < r) ∧ [A2] (∀r theta.  y = r * sin(theta)) ∧
  [A3] (∀r theta.  psi(r,theta) = sum (0..2) (∀n.  EL n [stream_uniform U y;
                         stream_doublet K theta r; stream_vortex gamma theta r]))
   ⇒ laplace_in_polar psi r theta
```

Assumptions A1-A2 are the same as those of Theorem 4.10. Assumption A3 provides the combination of the uniform, doublet and vortex flow solutions for the stream function, i.e., Equation (4.32). The verification of Theorem 4.12 is similar to that of Theorem 4.10.

Finally, we formally verify Equation (4.33) in HOL Light as follows:

Theorem 4.13. *Flow Past a Rotating Circular Cylinder for the Velocity Potential*

```
⊢thm ∀U K gamma x phi r theta.
  [A1] (∀r.  &0 < r) ∧ [A2] (∀r theta.  x = r * cos(theta)) ∧
  [A3] (∀r theta. phi(r,theta) = sum (0..2) (∀n.  EL n [velocity_uniform U x;
                       velocity_doublet K theta r; velocity_vortex gamma theta]))
   ⇒ laplace_vel_polar phi r theta
```

4.7.4 Summary and Discussion

In this chapter, we presented a higher-order logic formalization of the Laplace equation both in Cartesian and polar coordinates. We formally modeled fundamental flows such as the uniform, source/sink, doublet, and vortex flows in the HOL Light theorem prover. We then formally verified the validity of these exact potential flow solutions for the Laplace equation. Additionally, we presented the formal verification of the linearity of the Laplace operator, which is an essential property that enables the superposition of standard potential flows to construct more complex fluid dynamics. To illustrate the practical utility of our formalization, we formally verified several applications such as the Rankine oval, flow past a circular cylinder and flow past a rotating circular cylinder. More details about our formalizations and proofs are available in the related HOL Light proof script [100].

A significant contribution of this work is the development of the *first* formalization of potential flows, with broad applications in aerodynamics, particularly in airfoil theory. An important aspect of our work is the utilization of theorem proving into a domain that has been traditionally dominated by numerical techniques. While computational techniques often rely on approximations and can be prone to round-off errors and convergence issues, using proof assistants in this domain provide a more precise and reliable alternative. One of the main challenges of this work is its interdisciplinary nature, as it requires a deep understanding of aerodynamic principles, the integration of mathematics, and the meticulous process of interactive theorem proving. Another significant challenge is verifying exact potential flow solutions and their linear combinations governed by the Laplace equation. The proof process must establish the real derivatives of these solutions and their linear combinations. While traditional paper-and-pencil proofs can overlook trivial details, theorem proving demands a substantial amount of time due to the undecidable nature of higher-order logic and requires every detail to be meticulously provided to the computer. One of the benefits of this work is that it addresses these challenges by formalizing the core concepts of potential flow theory, allowing available results to be built upon to minimize user interaction. Furthermore, all of the verified theorems and lemmas are general, opening the door to future expansions. We believe that our work can be a significant step towards bridging the gap between theorem proving and the aerospace engineering communities, thereby enhancing its applicability in industrial settings.

In this chapter, we have provided a formal analysis of potential flows governed by the Laplace equation to gain insight into flow fields. In the following chapter, we address a different problem, focusing on the formal analysis of the behavior of voltage and current in electrical transmission lines, modeled by the telegrapher's and wave equations.

Chapter 5

Formalization of the Telegrapher's and Wave Equations

This chapter presents the formalization of the telegrapher's equations and their alternate representations, i.e., the wave equations as well as the formal verification of their analytical solutions in higher-order logic theorem proving. This includes the formal definitions of the telegrapher's equations and wave equations both in the time-and phasor-domains by proving the relationship between these equations in the phasor domain. Moreover, we develop the reasoning steps for the verification of the analytical solutions of these equations. In addition, we prove some important properties of special types of transmission lines that are lossless and distortionless. In order to demonstrate the utilization of our work, we formally analyze terminated, coupled and cascaded transmission lines.

As the wave equations can be derived from the telegrapher's equations to provide insights into electromagnetic wave propagation in electrical transmission lines, we present both these equations within a single chapter for comprehensive analysis. In the following section, we provide an overview of transmission line theory.

5.1 Brief Review of Transmission Line Theory

Transmission lines are a pivotal technology in electromagnetics because their theory is easy to understand and akin to circuit theory, making it accessible to most electrical engineers. The theory of transmission lines involves the propagation of voltage and current waves, which are described by coupled PDEs. In this context, voltage and current waves are represented as functions of two variables, in contrast to circuit theory where they are expressed as scalar quantities and functions of a single variable each. Therefore, the transmission line theory extends the scope of circuit theory by addressing its inadequacy in explaining wave phenomena. In fact, by adopting the transmission line theory, one can account for wave behavior and its associated physics, thereby bridging the theoretical gap left by circuit theory alone. Transmission lines are comprised of a minimum of two conductors that facilitate an efficient and a reliable transmission of information and energy. A two-conductor transmission line supports a *transverse electromagnetic* (TEM) wave [101], where the electric and magnetic fields are perpendicular to each other and transverse to the direction of propagation of waves along the transmission line. TEM waves have a fundamental property of establishing a distinct relationship between the electric **E** and the magnetic **H** fields, which are specifically related to the voltage V and current I, respectively as the following Maxwell's equations:

$$V = -\int_L \mathbf{E}.d\mathbf{l}, \tag{5.1}$$

$$I = \oint_L \mathbf{H}.d\mathbf{l} \tag{5.2}$$

The analysis of transmission lines can be made simpler by only focusing on the circuit quantities, V and I, rather than directly solving the complex line integral based Maxwell's equations (Equations (5.1) and (5.2)) and boundary conditions involving electric and magnetic fields ($\mathbf{E}$ and $\mathbf{H}$). In this regard, we employ an equivalent circuit in order to represent the transmission line's behavior. The development of an equivalent circuit model aims to simplify the complex electromagnetic interactions inherent in transmission lines by reducing them to a set of lumped elements that can be analyzed using the circuit theory. Once the equivalent circuit is constructed, the telegrapher's equations can be derived using circuit analysis techniques. The solutions of these differential equations allow us to understand the wave propagation (energy transmission) in the electrical transmission line. In the next section, we explain the derivation of the telegrapher's and wave equations from the corresponding circuit model.

5.2 Telegrapher's and Wave Equations

The transmission line or telegrapher's equations [102], used to model the propagation of electrical signals and energy along transmission lines. These equations were originally formulated by Oliver Heaviside in the 1880's and are an important example of PDEs in electrical engineering. Figure 5.1 [103] depicts an equivalent circuit model of a two-conductor transmission line. Here, R represents the line parameter resistance, whereas the other line parameters are the inductance L, the capacitance C, and the conductance G, which are specified per unit length (Δz). Moreover, $V(z,t)$ and $V(z+\Delta z,t)$ are the input and output voltages, respectively. Similarly, $I(z,t)$ and $I(z+\Delta z,t)$ are the input and output currents, respectively. Moreover, both voltage and current are functions of space and time.

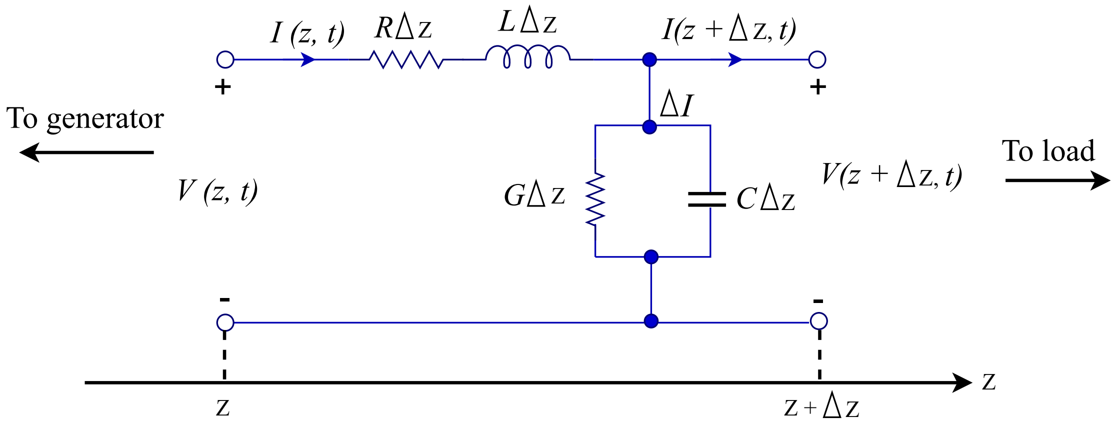


Figure 5.1: Equivalent Circuit of Two-Conductor Transmission Line

5.2.1 Telegrapher's and Wave Equations in Time-Domain

The Law of Conservation of Energy, attributed to Kirchhoff, asserts that there is no loss of voltage throughout a closed loop or circuit; instead, one returns to the initial point within the circuit and, consequently, to the same initial electric potential. Hence, any reductions in voltage within the circuit must balance out with the voltage sources encountered along the same route. By applying the Kirchhoff's voltage law to the circuit of two-conductor transmission line of Figure 5.1, we get the following equations [5]:

$$V(z+\Delta z,t) - V(z,t) = -R\Delta z I(z,t) - L\Delta z \frac{\partial I(z,t)}{\partial t} \tag{5.3}$$

Next, dividing Equation (5.3) by Δz and applying the limit $\Delta z \to 0$, we obtain:

$$\lim_{\Delta z \to 0} \frac{V(z+\Delta z,t) - V(z,t)}{\Delta z} = \frac{-R\Delta z I(z,t)}{\Delta z} - L\frac{\Delta z}{\Delta z}\frac{\partial I(z,t)}{\partial t}$$

Finally, by using the definition of the partial derivative, we get:

$$\frac{\partial V(z,t)}{\partial z} = -RI(z,t) - L\frac{\partial I(z,t)}{\partial t} \tag{5.4}$$

Similarly, by applying the Kirchhoff's current law to the circuit, we find [5]:

$$I(z+\Delta z,t) - I(z,t) = -G\Delta z V(z+\Delta z,t) - C\Delta z\frac{\partial V(z+\Delta z,t)}{\partial t} \tag{5.5}$$

Next, dividing Equation (5.5) by Δz and using the definition of the partial derivative, we get:

$$\frac{\partial I(z,t)}{\partial z} = -GV(z,t) - C\frac{\partial V(z,t)}{\partial t} \tag{5.6}$$

Equations (5.4) and (5.6) are known as the *telegrapher's equations* that provide a time-domain relationship between the voltage and current in any transmission line.

Now, we can combine the telegrapher's equations (Equations (5.4) and (5.6)) to obtain their alternate representations that are commonly known as the *wave equations*, which are more practical to use and provide some additional physical insights and are mathematically expressed as follows:

$$\frac{\partial^2 V(z,t)}{\partial z^2} - LC\frac{\partial^2 V(z,t)}{\partial t^2} = (RC+GL)\frac{\partial V(z,t)}{\partial t} + RGV(z,t) \tag{5.7}$$

$$\frac{\partial^2 I(z,t)}{\partial z^2} - LC\frac{\partial^2 I(z,t)}{\partial t^2} = (RC+GL)\frac{\partial I(z,t)}{\partial t} + RGI(z,t) \tag{5.8}$$

where $\frac{\partial^2}{\partial z^2}$ and $\frac{\partial^2}{\partial t^2}$ capture the second-order partial derivative with respect to z and t, respectively.

Next, we express the space-time voltage and current functions as *phasors* in order to reduce the PDEs to ODEs, which will greatly facilitate the derivation of the general solutions of these equations.

The relationship between the space-time voltage and current functions and their phasors can be mathematically expressed as follows [104]:

$$V(z,t) = \mathscr{R}e\{V(z)e^{j\omega t}\}$$

$$I(z,t) = \mathscr{R}e\{I(z)e^{j\omega t}\}$$

where $V(z)$ and $I(z)$ are the phasor components corresponding to $V(z,t)$ and $I(z,t)$, respectively.

5.2.2 Telegrapher's and Wave Equations in Phasor Domain

The principal advantage of the phasor representation of the telegrapher's equations over the time-domain versions is that we no longer need the derivatives with respect to time and are left with the derivatives with respect to distance only. This considerably simplifies the corresponding equations. For instance, the sinusoidally time-varying case, the telegrapher's equations (Equations (5.4) and (5.6)) can be rewritten in terms of phasor quantities by replacing $\dfrac{\partial}{\partial t}$ with $j\omega$. We can derive the telegrapher equation for voltage from Equation (5.4) as follows:

$$\frac{\partial V(z,t)}{\partial z} = -RI(z,t) - L\frac{\partial I(z,t)}{\partial t}$$

$$\frac{\partial}{\partial z}[\underbrace{\mathscr{R}e\{V(z)e^{j\omega t}\}}_{V(z,t)}] = -R[\underbrace{\mathscr{R}e\{I(z)e^{j\omega t}\}}_{I(z,t)}] - L\frac{\partial}{\partial t}[\underbrace{\mathscr{R}e\{I(z)e^{j\omega t}\}}_{I(z,t)}]$$

$$\mathscr{R}e\Big\{e^{j\omega t}\frac{dV(z)}{dz}\Big\} = \mathscr{R}e\{-RI(z)e^{jwt} - L(j\omega)e^{j\omega t}I(z)\}$$

$$\frac{dV(z)}{dz} = (-R - j\omega L)I(z)$$

From the above, we can rewrite the telegrapher's equations for voltage as:

$$\frac{dV(z)}{dz} + (R + j\omega L)I(z) = 0 \tag{5.9}$$

We can also derive the following Equation (5.10) from Equation (5.6) in a similar manner

$$\frac{dI(z)}{dz} + (G + j\omega C)V(z) = 0 \tag{5.10}$$

Here, Equations (5.9) and (5.10) are ODEs due to the fact that V and I are functions of the single variable z. Equation (5.9) indicates that the rate of change of the phasor voltage along the transmission line, as a function of position z, is equal to the series impedance of the line per unit length multiplied by the phasor current. Similarly, Equation (5.10) states that the rate of change of phasor current along the transmission line, as a function of position z, is equal to the shunt admittance of the line per unit length multiplied by the phasor voltage.

A limitation in using the above form of the telegrapher's equations (Equations (5.9) and (5.10)) is that we need to solve each of them for both voltage and current. To reduce such overhead, we can write the telegrapher's equations using one function (*V(z)* or *I(z)*) as equivalent wave equations. To do this, we first take the derivative of Equation (5.9) with respect to z:

$$\frac{d}{dz}\left\{\frac{dV(z)}{dz} = -(R + j\omega L)I(z)\right\}$$

which can be written as:

$$\frac{d^2V(z)}{dz^2} = -(R + j\omega L)\frac{dI(z)}{dz} \tag{5.11}$$

Next, we substitute Equation (5.10) in Equation (5.11), to obtain the following equation that involves only *V(z)*:

$$\frac{d^2V(z)}{dz^2} = \gamma^2 V(z) \tag{5.12}$$

γ is the complex propagation constant and is mathematically expressed as:

$$\gamma = \alpha + j\beta = \sqrt{(R + j\omega L)(G + j\omega C)}. \tag{5.13}$$

where α is the attenuation coefficient and β is the phase coefficient and both are mathematically expressed as:

$$\alpha = \mathscr{Re}(\gamma) = \mathscr{Re}\{\sqrt{(R + j\omega L)(G + j\omega C)}\}$$

$$\beta = \mathscr{Im}(\gamma) = \mathscr{Im}\{\sqrt{(R + j\omega L)(G + j\omega C)}\}$$

In a similar manner, we derive the second wave equation by taking the derivative of Equation (5.10) and substituting Equation (5.9) in the resultant equation:

$$\frac{d^2I(z)}{dz^2} = \gamma^2 I(z) \tag{5.14}$$

We can alternatively represent the wave equations (Equations (5.12) and (5.14)) as:

$$\frac{d^2V(z)}{dz^2} - \gamma^2 V(z) = 0 \tag{5.15}$$

$$\frac{d^2 I(z)}{dz^2} - \gamma^2 I(z) = 0 \tag{5.16}$$

By employing phasor representation, we have reduced the time-domain telegrapher's and wave equations to a set of ODEs. These ODEs are then solved in the phasor domain, enabling us to obtain the solutions for the original time-domain PDEs. In the following section, we present the details for the formal analysis of the telegrapher's and wave equations.

5.3 Proposed Methodology

The proposed approach for formally analyzing the telegrapher's equations and their derived form (the wave equations) using higher-order logic theorem proving is depicted in Figure 5.2. The input to this framework, represented by a rectangle, consist of the mathematical formulation of the telegrapher's and wave equations. The first step of our proposed approach is to formalize the telegrapher's and the wave equations in time and phasor domains. The formalization of these equations requires HOL Light's libraries of multivariate calculus, such as differential, transcendental and complex vectors. The next step is to establish theorems that enable the formal verification of solutions for these equations by leveraging the advantages of the phasor domain representation of these equations. Moreover, the relationship between the telegrapher's and the wave equations in the phasor domain is formally verified using these theorems. Subsequently, we use the solutions in the phasor domain to verify the PDEs by establishing a relationship between the corresponding functions in the phasor and time-domains.

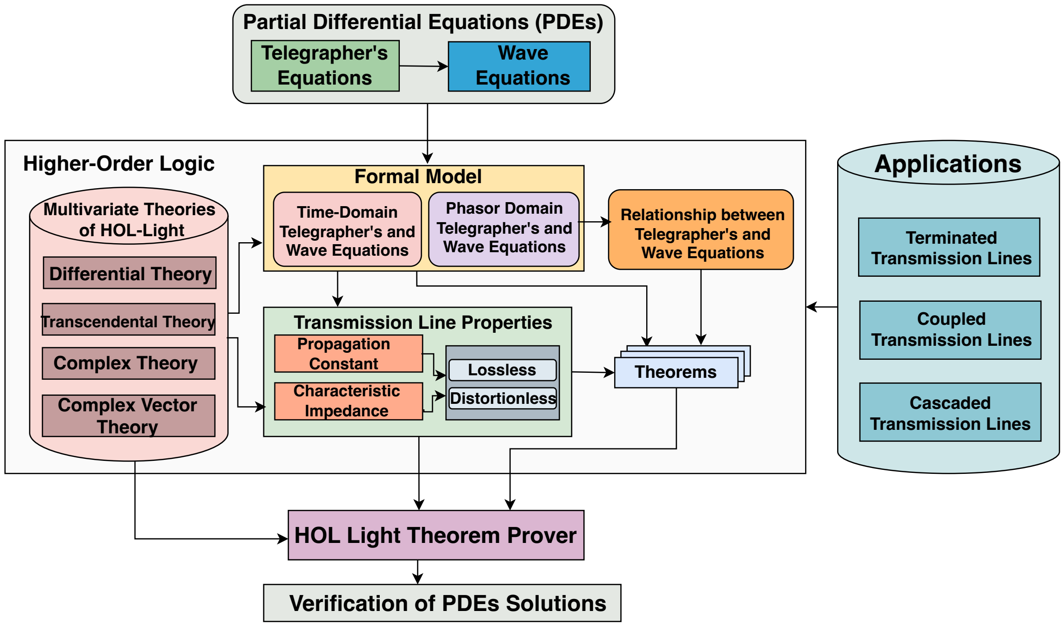


Figure 5.2: Proposed Methodology for the Telegrapher's and Wave Equations

All theorems of the proposed framework of the telegrapher's equations are verified in HOL Light in a generic way in order to obtain general (universally quantified) solutions of the related PDEs. The next step is to verify some important properties of transmission lines, such as the propagation constant and the characteristic impedance specifically focusing on the case of lossless and distortionless lines as shown in Figure 5.2. Moreover, in order to demonstrate the practical effectiveness of the proposed formalization, we conduct a formal analysis of terminated, coupled and cascaded transmission lines.

In the following section, we present the formalization of the telegrapher's and wave equations in the HOL Light proof assistant.

5.4 Formalization of the Telegrapher's and Wave Equations

In this section, we present the formalization of the telegrapher's equations and the wave equations for voltage and current in HOL Light in both time- and-phasor domains.

5.4.1 Formalization of the Telegrapher's and Wave Equations in Time Domain

We first start the formalization of the telegrapher's equations (Equations (5.4) and (5.6)) in time domain as follows:

Definition 5.1. *Telegrapher's Equation for Voltage*

```
⊢def ∀V I R L z t.  telegraph_equation_voltage V I R L z t ⇔
  (complex_derivative (λz. V z t) z) =
    --(Cx L * complex_derivative (λt. I z t) t - Cx R * (I z t))
```

Definition 5.2. *Telegrapher's Equation for Current*

```
⊢def ∀V I G C z t.  telegraph_equation_current V I G C z t ⇔
  (complex_derivative (λz. I z t) z) =
    --(Cx C * complex_derivative (λt. V z t) t) - Cx G * (V z t)
```

where `telegraph_equation_voltage` and `telegraph_equation_current` mainly accept the functions `V` and `I` of type $\mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}$, representing the voltage and current, respectively, and return the corresponding telegrapher's equations. The variables `R:`$\mathbb{R}$, `L:`$\mathbb{R}$, `G:`$\mathbb{R}$ `C:`$\mathbb{R}$, `z:`$\mathbb{C}$, and `t:`$\mathbb{C}$ represent the resistance, inductance, conductance, capacitance, the spatial coordinate and the time variable, respectively.

It is important to note that we use `complex_derivative` to formalize the time-domain PDEs due to the involvement of the phasor domain representations of the voltage and current functions in the analysis. Since a phasor domain representation of a function is a vector in complex plane with some magnitude and angle, the variables `z` and `t` are considered as complex numbers for convenience and the corresponding voltages and currents equations equally hold under this choice.

To model the wave equations for voltage and current, we need the transmission line constants, such as R, L, G and C. Therefore, we use the type abbreviation in HOL Light providing a compact representation of these constants as follows:

Definition 5.3. *Transmission Line Constants*

```
new_type_abbrev ("R",`:ℝ`)

new_type_abbrev ("L",`:ℝ`)

new_type_abbrev ("G",`:ℝ`)

new_type_abbrev ("C",`:ℝ`)

new_type_abbrev ("trans_line_const",`:R # L # G # C`)
```

Now, we formalize the wave equations for both voltage (Equation (5.7)) and current (Equation (5.8)) in time-domain as follows:

Definition 5.4. *Wave Equation for Voltage*

```
⊢def ∀V R L G C z t.
wave_voltage_equation V ((R,L,G,C):trans_line_const) z t ⇔
 higher_complex_derivative 2 (λz. V z t) z -
   Cx L * Cx C * (higher_complex_derivative 2 (λt. V z t) t =
    (Cx R * Cx C + Cx G * Cx L) * (complex_derivative (λt. V z t) t) +
                                                     Cx R * Cx G * (V z t))
```

Definition 5.5. *Wave Equation for Current*

```
⊢def ∀I R L G C z t.
 wave_current_equation I ((R,L,G,C):trans_line_const) z t ⇔
 higher_complex_derivative 2 (λz. I z t) z -
   Cx L * Cx C (higher_complex_derivative 2 (λt. I z t) t =
    (Cx R * Cx C + Cx G * Cx L) * (complex_derivative (λt. I z t) t) +
                                                   Cx R * Cx G * (I z t))
```

5.4.2 Formalization of the Telegrapher's and Wave Equations in Phasor Domain

Now, we formalize the telegrapher's equation in the phasor domain for voltage (Equation (5.9)) as:

Definition 5.6. *Telegrapher's Equation*

```
⊢def ∀V I R L w z.  telegraph_equation_phasor_voltage V I R L w z ⇔
                                    telegraph_voltage V I R L w z = Cx(&0)
```

where `telegraph_equation_phasor_voltage` accepts the complex functions `V`:$\mathbb{C} \rightarrow \mathbb{C}$ and `I`:$\mathbb{C} \rightarrow \mathbb{C}$, the line parameters `R`:$\mathbb{R}$ and `L`:$\mathbb{R}$, the angular frequency ω:$\mathbb{R}$, the spatial coordinate `z`:$\mathbb{C}$, and returns the corresponding telegrapher's equation. Here, the function `telegraph_voltage` models the left-hand side of Equation (5.9), and is formalized as follows:

Definition 5.7. *Left-Hand Side of Equation (5.9)*

```
⊢def ∀V I R L w z.  telegraph_voltage V I R L w z =
    complex_derivative (λz. V(z)) z + (Cx R + ii * Cx w * Cx L) * I(z)
```

Similarly, we formalize Equation (5.10) in HOL Light as follows:

Definition 5.8. *Telegrapher's Equation*

```
⊢def ∀V I G C w z.  telegraph_equation_phasor_current V I G C w z ⇔
                                    telegraph_current V I G C w z = Cx(&0)
```

with

Definition 5.9. *Left-Hand Side of Equation (5.10)*

```
⊢def ∀V I G C w z.  telegraph_current V I G C w z =
    complex_derivative (λz. I(z)) z + (Cx G + ii * Cx w * Cx C) * V(z)
```

where `telegraph_current` models the left-hand side of Equation (5.10).

Now, to verify a relationship between the telegrapher's and wave equations for voltage and current in the phasor domain, we first formalize the propagation constant in HOL Light as follows:

Definition 5.10. *Propagation Constant*

```
⊢def ∀R L G C w.
 propagation_constant ((R,L,G,C):trans_line_const) w =
   csqrt ((Cx R + ii * Cx w * Cx L) * (Cx G + ii * Cx w * Cx C))
```

The function `propagation_constant` accepts the transmission line parameters `R`, `L`, `G`, `C` and angular frequency ω, and returns the corresponding function.

The wave equations (Equations (5.15) and (5.16)) in higher-order logic are formalized as:

Definition 5.11. *Wave Equation for Voltage*

```
⊢def ∀V tlc w z.  wave_equation_phasor_voltage V z tlc w ⇔
                         wave_voltage V z tlc w = Cx(&0)
```

where the function `wave_voltage` defined as:

Definition 5.12. *Left-Hand Side of Equation (5.15)*

$\vdash_{def}$ ∀V tlc w z.

```
wave_voltage V z tlc w = higher_complex_derivative 2 (λz. V(z)) z -
                          (propagation_constant tlc w)^2 * V(z)
```

Definition 5.13. *Wave Equation for Current*

$\vdash_{def}$ ∀I tlc w z. wave_equation_phasor_current I z tlc w z ⇔

```
                 wave_current I z tlc w = Cx(&0)
```

where the function wave_current defined as:

Definition 5.14. *Left-Hand Side of Equation (5.16)*

$\vdash_{def}$ ∀I tlc w z.

```
wave_current I z tlc w = higher_complex_derivative 2 (λz. I(z)) z -
                          (propagation_constant tlc w)^2 * I(z)
```

5.4.3 Relationship between Telegrapher's and Wave Equations in Phasor Domain

In this section, we formally verify the relationship between the telegrapher's and wave equations for voltage and current in the phasor domain as the following HOL Light theorems:

Theorem 5.1. *Relationship between Telegrapher's and Wave Equations for Voltage*

$\vdash_{thm}$ ∀V I R L G C w z.

```
let tlc = ((R,L,G,C):trans_line_const) in
[A1] (λz. complex_derivative (λz. V z) z) complex_differentiable at z ∧
[A2] I complex_differentiable at z ∧
[A3] telegraph_current V I G C w z = Cx(&0)
 ⇒ complex_derivative (λz. telegraph_voltage V I R L w z) z =
                          wave_voltage V z tlc w
```

Assumption `A1` ensures that the first-order derivative of the function `V` is differentiable at `z`. Assumption `A2` asserts the differentiability of the function `I` at `z`. Assumption `A3` provides the telegrapher's equation for current, i.e., Equation (5.10). The proof of Theorem 5.1 is mainly based on the definitions of the telegrapher's and wave equations and some classical properties of the complex derivative along with some complex arithmetic reasoning. Similarly, we formally verify this relationship for the current in phasor domain.

Theorem 5.2. *Relationship between Telegrapher's and Wave Equations for Current*

```
⊢thm ∀V I R L G C w z.
let tlc = ((R,L,G,C):trans_line_const) in
[A1] (λz. complex_derivative (λz. I z) z) complex_differentiable at z ∧
[A2] V complex_differentiable at z ∧
[A3] telegraph_voltage V I R L w z = Cx(&0)
  ⇒ complex_derivative (λz. telegraph_current V I G C w z) z =
                                          wave_current I z tlc w
```

The verification of Theorem 5.2 is very similar to that of Theorem 5.1. More details about their verification can be found at [105].

5.5 Formal Verification of the Solutions of the Telegrapher's and Wave Equations

Analyzing transmission lines is mainly based on finding out solutions of these PDE based telegrapher's and wave equations that are further used to analyze various aspects of signal propagation, such as attenuation, distortion, reflection, and dispersion along the transmission line. One of the examples is to understand the behavior of high-frequency signals, where the distributed parameters of the transmission line significantly affect the signal integrity. In this section, we formally verify the correctness of the analytical solutions of the telegrapher's equations and their alternate representations, i,e., wave equations in the phasor domain pertaining to sinusoidal steady state and in the time-domain that are concerned with arbitrary variations over time.

5.5.1 Verification of the Solutions in Phasor Domain

We can mathematically express the general solutions of the wave equations (and thus the telegrapher's equations) (Equations (5.15) and (5.16)) as follows:

$$V(z) = V^{+}(z) + V^{-}(z) = V_0^{+}e^{-\gamma z} + V_0^{-}e^{\gamma z} \tag{5.17}$$

$$I(z) = I^{+}(z) + I^{-}(z) = I_0^{+}e^{-\gamma z} + I_0^{-}e^{\gamma z} \tag{5.18}$$

Here, V_0^{+}, V_0^{-}, I_0^{+}, I_0^{-} are the complex constants that can be determined by boundary conditions. Similarly, the transmission line voltage $V^{+}(z)$ and current $I^{+}(z)$ represent the forward-going waves (propagating in the $+z$ direction) and voltage $V^{-}(z)$ and current $I^{-}(z)$ are the backward-going waves (propagating in the $-z$ direction).
If we insert the solution for $V(z)$ in Equation (5.9), we get:

$$\frac{dV(z)}{dz} = -\gamma V_0^{+}e^{-\gamma z} + \gamma V_0^{-}e^{\gamma z} = -(R + j\omega L)I(z) \tag{5.19}$$

Next, we rearrange the above equation to obtain the current *I(z)*:

$$I(z) = \frac{\gamma}{R + j\omega L}(V_0^{+}e^{-\gamma z} - V_0^{-}e^{\gamma z}) \tag{5.20}$$

Note that both expressions (Equations (5.18) and (5.20)) for the current are the same. The characteristic impedance, which is the ratio of the line voltage and current, is an important characteristic of transmission line and can be mathematically expressed as follows:

$$Z_0 = \frac{V_0^+}{I_0^+} = \frac{-V_0^-}{I_0^-} = \sqrt{\frac{R + j\omega L}{G + j\omega C}} = \frac{R + j\omega L}{\gamma} = R_0 + jX_0 \tag{5.21}$$

where R_0 and X_0 are the real and imaginary parts of Z_0. The characteristic impedance Z_0 and the propagation constant γ are two important properties of the transmission line due to their direct dependence on the line parameters R, L, G, C and the phasor of the operation.

Next, we define the characteristic impedance in HOL Light as follows:

Definition 5.15. *Characteristic Impedance*

$\vdash_{def}$ ∀R L G C w. characteristic_impedance (R,L,G,C) w =

(let tlc = ((R,L,G,C):trans_line_const) in $\dfrac{\texttt{Cx R + ii * Cx w * Cx L}}{\texttt{propagation\_constant tlc w}}$

The next step is to formalize the general solutions (Equations (5.17) and (5.20)) in HOL Light:

Definition 5.16. *Wave Solution for Voltage*

```
⊢def ∀V1 V2 tlc w z.  wave_solution_voltage_phasor V1 V2 tlc w z =
      V1 * cexp(--(propagation_constant tlc w) * z) +
        V2 * cexp((propagation_constant tlc w) * z)
```

where `V1` and `V2` in the formalization refer to the complex constants V_0^+ and V_0^- in Equation (5.17), respectively. The parameters `w` and `z` represent the angular frequency and the spatial coordinate, respectively.

Definition 5.17. *Wave Solution for Current*

$\vdash_{def}$ ∀V1 V2 tlc w z. wave_solution_current_phasor V1 V2 tlc w z =

$$\frac{\texttt{Cx(\&1)}}{\texttt{characteristic\_impedance tlc w}} *$$

```
    (V1 * cexp(--(propagation_constant tlc w) * z) -
        V2 * cexp((propagation_constant tlc w) * z))
```

Next, we formally verify the general solutions (Equations (5.18) and (5.20)) of the wave equations for voltage and current, (represented by Equations (5.15) and (5.16)), in the HOL Light theorem prover as follows:

Theorem 5.3. *Correctness of the Solution of the Wave Equation for Voltage*

$\vdash_{thm}$ ∀V1 V2 V R L G C w z.

```
  let tlc = ((R,L,G,C):trans_line_const) in
   wave_equation_voltage_phasor
     (λz. wave_solution_voltage_phasor V1 V2 tlc w z) V tlc w
```

Theorem 5.4. *Correctness of the Solution of the Wave Equation for Current*

$\vdash_{thm}$ ∀V1 V2 I R L G C w z.

```
  let tlc = ((R,L,G,C):trans_line_const) in
   wave_equation_current_phasor
     (λz. wave_solution_current_phasor V1 V2 tlc w z) I tlc w
```

The verification of Theorems 5.3 and 5.4 is mainly based on four lemmas about the complex differentiation of the solutions, given in Table 5.1.

Since there exists a relationship between the telegrapher's and wave equations, as proven in Section 5.4, the solutions of the wave equations also satisfy the telegrapher's equations. We now formally verify the general solution (Equation 5.17) of the telegrapher's equation for the voltage, i.e. Equation (5.9).

Table 5.1: Lemmas of the Derivatives of General Solutions in Phasor Domain

Mathematical Form	Formalized Form
$\frac{dV(z)}{dz} = -\gamma V_1 e^{-\gamma z} + \gamma V_2 e^{\gamma z}$	**Lemma 1 (First-Order Derivative of General Solution for Voltage):** ∀V1 V2 R L G C w z. let tlc = ((R,L,G,C):trans_line_const) in complex_derivative (λz. wave_solution_voltage_phasor V1 V2 tlc w z) z = V1 * (--(propagation_constant tlc w)) * cexp (--(propagation_constant tlc w) * z) + V2 * (propagation_constant tlc w) * cexp ((propagation_constant tlc w) * z)
$\frac{d^2V(z)}{dz^2} = \gamma^2 V_1 e^{-\gamma z} + \gamma^2 V_2 e^{\gamma z}$	**Lemma 2 (Second-Order Derivative of General Solution for Voltage):** ∀V1 V2 R L G C w z. let tlc = ((R,L,G,C):trans_line_const) in higher_complex_derivative 2 (λz. wave_solution_voltage_phasor V1 V2 tlc w z) z = V1 * (propagation_constant tlc w)^2 * cexp (--(propagation_constant tlc w) * z) + V2 * (propagation_constant tlc w)^2 * cexp ((propagation_constant tlc w) * z
$\frac{dI(z)}{dz} = \frac{1}{Z_0}(-\gamma V_1 e^{-\gamma z} - \gamma V_2 e^{\gamma z})$	**Lemma 3 (First-Order Derivative of General Solution for Current):** ∀V1 V2 R L G C w z. let tlc = ((R,L,G,C):trans_line_const) in complex_derivative (λz. wave_solution_current_phasor V1 V2 tlc w z) z = Cx(&1) / characteristic_impedance tlc w * (V1 * (--propagation_constant tlc w) * cexp (--(propagation_constant tlc w) * z) - V2 * (propagation_constant tlc w) * cexp ((propagation_constant tlc w) * z))
$\frac{d^2I(z)}{dz^2} = \frac{1}{Z_0}(\gamma^2 V_1 e^{-\gamma z} - \gamma^2 V_2 e^{\gamma z})$	**Lemma 4 (Second-Order Derivative of General Solution for Current):** ∀V1 V2 R L G C w z. let tlc = ((R,L,G,C):trans_line_const) in higher_complex_derivative 2 (λz. wave_solution_current_phasor V1 V2 tlc w z) z = Cx(&1) / characteristic_impedance tlc w * (V1 * (propagation_constant tlc w)^2 * cexp (--(propagation_constant tlc w) * z) - V2 * (propagation_constant tlc w)^2 * cexp ((propagation_constant tlc w) * z))

Theorem 5.5. *Correctness of the Solution of the Telegrapher's Equation for Voltage*

$\vdash_{thm}$ ∀V1 V2 V I R L G C w.

let tlc = ((R,L,G,C):trans_line_const) in

[A1] Cx R + ii * Cx w * Cx L ≠ Cx(&0) ∧

```
[A2] (∀z. V z = wave_solution_voltage_phasor V1 V2 tlc w z) ∧
[A3] (∀z. I z = wave_solution_current_phasor V1 V2 tlc w z)
  ⇒ telegraph_equation_phasor_voltage V I R L w z
```

Assumption A1 ensures that expression $R+j\omega L$ is not equal to zero. Assumptions A2 and A3 provide the solutions of the wave equations for the voltage and the current, respectively. The verification of the above theorem is based on the properties of the complex differentiation along with some complex arithmetic reasoning.

Theorem 5.6. *Correctness of the Solution of the Telegrapher's Equation for Current*

```
⊢thm ∀V1 V2 V I R L G C w.
let tlc = ((R,L,G,C):trans_line_const) in
[A1] Cx R + ii * Cx w * Cx L ≠ Cx(&0) ∧
[A2] (∀z. V z = wave_solution_voltage_phasor V1 V2 tlc w z) ∧
[A3] (∀z. I z = wave_solution_current_phasor V1 V2 tlc w z)
  ⇒ telegraph_equation_phasor_current V I G C w z
```

Assumptions A1-A3 are the same as those of Theorem 5.5. The conclusion of the theorem provides the telegrapher's equation for the current, i.e., Equation 5.10. The verification of the above theorem is similar to that of Theorem 5.5.

5.5.2 Verification of Properties of Transmission Lines

A transmission line is characterized by two essential properties, namely its propagation constant γ and characteristic impedance Z_0. These properties are specified by the angular frequency ω and the line parameters R, L, G and C. Understanding and optimizing the transmission line characteristics help engineers and designers to achieve efficient signal transmission, maintain signal integrity, and ensure the reliable operation of these systems. In the following, we formally verify these transmission line properties for the case of lossless and distortionless lines.

5.5.2.1 Lossless Line

The main purpose of a transmission line is to facilitate the transmission of information between distant locations with minimal signal degradation that can be achieved by reducing the signal loss. This is one of the crucial requirements in the construction of an efficient and a reliable transmission line. In the case of a lossless transmission line, the elements R (resistance) and G (conductance) can be considered as negligible or effectively zero:

$$R = G = 0$$

The characteristic impedance of a lossless transmission line can now be expressed in a simplified form by using the above values of R and G in Equation (5.21) as:

$$Z_0 = \sqrt{\frac{j\omega L}{j\omega C}} = \sqrt{\frac{L}{C}}$$

We now formally verify the characteristic impedance for a lossless line as the following HOL Light theorem:

Theorem 5.7. *Characteristic Impedance for a Lossless Line*

$\vdash_{thm}$ ∀R L G C w.

```
let tlc = ((R,L,G,C):trans_line_const) in
[A1] w > &0 ∧ [A2] L > &0 ∧ [A3] C > &0
[A4] R = &0 ∧ [A5] G = &0
[A6] Cx G + ii * Cx w * Cx C ≠ Cx(&0)
[A7] csqrt(Cx L * Cx C) ≠ Cx(&0)
```

$\Rightarrow$ `characteristic_impedance tlc w =` $\dfrac{\texttt{csqrt(Cx(L)} * \texttt{Cx(C))}}{\texttt{Cx(C)}}$

Assumptions `A1-A3` guarantee that the angular frequency ω, the line parameters `L` and `C` are positive real values. Assumptions `A4-A5` assert that the line parameters `R` and `G` are equal to zero, which is an assumption for a lossless transmission line. Assumptions A6 and A7 ensure that the expressions $G + j\omega C$ and $\sqrt{LC}$ are not equal to zero. The proof of the above theorem is mainly based on complex arithmetic reasoning.

Similarly, the attenuation and phase constants expressed in Equation (5.13) becomes:

$$\alpha = 0 \tag{5.22}$$

$$\beta = \omega\sqrt{LC} \tag{5.23}$$

This implies that the transmission line has no signal attenuation, and as a result, the propagation constant can be represented by a purely imaginary number:

$$\gamma = j\beta = j\omega\sqrt{LC}$$

Next, we verify the attenuation and phase constants (Equations (5.22) and (5.23)) in HOL Light as follows:

Theorem 5.8. *Attenuation Constant for a Lossless Line*

```
⊢thm ∀R L G C w.
let tlc = ((R,L,G,C):trans_line_const) in
[A1] w > &0 ∧ [A2] L > &0 ∧ [A3] C > &0
[A4] R = &0 ∧ [A5] G = &0
  ⇒ Re(propagation_constant tlc w) = 0
```

Asummptions A1-A5 are the same as those of Theorem 5.7. The proof of this theorem requires complex arithmetic simplification.

Theorem 5.9. *Phase Constant for a Lossless Line*

```
⊢thm ∀R L G C w.
let tlc = ((R,L,G,C):trans_line_const) in
[A1] w > &0 ≠ Cx(&0) ∧ [A2] L > &0 ∧ [A3] C > &0
[A4] R = &0 ∧ [A5] G = &0
  ⇒ Im(propagation_constant tlc w) = w * sqrt(L * C)
```

Asummptions A1-A5 are the same as those of Theorem 5.8. The proof of this theorem is also similar to that of Theorem 5.8.

5.5.2.2 Distortionless Line

A distortionless line refers to a transmission medium characterized by an attenuation constant α that exhibits no variation with changes in frequency while the phase constant β is linearly dependent on frequency.

For a distortionless transmission line, the elements R and G are related as:

$$\frac{R}{L} = \frac{G}{C}$$

Now, the characteristic impedance of the transmission line is expressed as:

$$Z_0 = \sqrt{\frac{R(1 + j\omega L/R)}{R(1 + j\omega C/G)}} = \sqrt{\frac{R}{G}} = \sqrt{\frac{L}{C}}$$

Now, formally verify the characteristic impedance for a distortionless line in HOL Light as follows:

Theorem 5.10. *Characteristic Impedance*

$\vdash_{thm}$ ∀R L G C w.

let tlc = ((R,L,G,C):trans_line_const) in

[A1] L > &0 ∧ [A2] C > &0 [A2] R > &0

[A4] G > &0 ∧ [A5] Cx G + ii * Cx w * Cx C ≠ Cx(&0)

[A6] $\frac{\texttt{R}}{\texttt{L}} = \frac{\texttt{G}}{\texttt{C}}$

⇒ characteristic_impedance tlc w = $\frac{\texttt{csqrt(Cx(L)} * \texttt{Cx(C))}}{\texttt{Cx(C)}}$

Assumptions A1-A4 assert that the transmission line parameters are positive. Assumption A5 indicates that the expression $G + j\omega C$ is not equal to zero. Assumption

A6 is an assumption for a distortionless line. The proof of this theorem is primarily based on complex arithmetic reasoning and complex arithmetic simplification.

The propagation constant (Equation (5.13)) becomes:

$$\gamma = \sqrt{RG\left(1+\frac{j\omega L}{R}\right)\left(1+\frac{j\omega C}{G}\right)}$$

$$\gamma = \sqrt{RG}\left(1+\frac{j\omega C}{G}\right) = \alpha + j\beta$$

or

$$\alpha = \sqrt{RG}, \qquad \beta = \omega\sqrt{LC} \tag{5.24}$$

We can see that the attenuation constant α is independent of the frequency, whereas β is a linear function of frequency.

Next, we formally verify the attenuation and phase constants as the following HOL Light theorems, respectively:

Theorem 5.11. *Attenuation Constant for Distortionless Line*

```
⊢thm ∀R L G C w.
let tlc = ((R,L,G,C):trans_line_const) in
[A1] L > &0 ∧ [A2] R > &0 [A3] G > &0 [A4] R/L = G/C
  ⇒ Re(propagation_constant tlc w) = sqrt(R * G)
```

Assumptions A1-A4 are the same as those of Theorem 5.10. The verification of the above theorem is similar to that of Theorem 5.10.

Theorem 5.12. *Phase Constant for Distortionless Line*

```
⊢thm ∀R L G C w.  let tlc = ((R,L,G,C):trans_line_const) in
[A1] L > &0 ∧ [A2] R > &0 [A3] G > &0
[A4] C > &0 ∧ [A5] R/L = G/C
  ⇒ Im(propagation_constant tlc w) = w * sqrt(L * C)
```

In the following section, we verify the general solutions of the time-domain PDEs by considering a lossless line, where we assume both resistance R and conductance G to be zero.

5.5.3 Verification of the PDEs Solutions in Time-Domain

It is useful to examine the complete time functions for understanding the nature of the voltage and current within a transmission line. We can find the corresponding time-domain expressions for voltage and current (solution in the time-domain) on the line by multiplying the phasor of the voltage and current with the harmonic time variation term e^{jwt} and taking its real part as follows:

$$V(z,t) = \mathscr{R}e\{V(z)e^{j\omega t}\} \tag{5.25}$$

$$I(z,t) = \mathscr{R}e\{I(z)e^{j\omega t}\} \tag{5.26}$$

Next, we use Equation (5.17) in the time-domain solution (Equation (5.25)) and get:

$$V(z,t) = \mathscr{R}e\{(V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z})e^{j\omega t}\}$$

$$V(z,t) = \mathscr{R}e\{V_0^+ e^{-\gamma z} e^{j\omega t} + V_0^- e^{\gamma z} e^{j\omega t}\}$$

By splitting the propagation constant in real and imaginary parts, i.e., $\gamma = \alpha + j\beta$, we can write the above equation for voltage as follows:

$$V(z,t) = \mathscr{R}e\{V_0^+ e^{-(\alpha+j\beta)z} e^{j\omega t} + V_0^- e^{(\alpha+j\beta)z} e^{j\omega t}\}$$

We know that α is equal to zero for a lossless transmission line. Thus, we get:

$$V(z,t) = \mathscr{R}e\{V_0^+ e^{j(\omega t-\beta z)} + V_0^- e^{j(\omega t+\beta z)}\} \tag{5.27}$$

After applying Euler's formula to the above equation and taking the real part of the solution, we have:

$$V(z,t) = V_0^+ \cos(\omega t - \beta z) + V_0^- \cos(\omega t + \beta z) \tag{5.28}$$

where we assume V_0^+ and V_0^- to be real.

Using Definition 5.16, we formalize the general solution (Equation (5.25)) in the time-domain for voltage as follows:

Definition 5.18. *General Solution for Voltage in Time-Domain*

```
⊢def ∀V1 V2 tlc w z t.
wave_solution_voltage_time V1 V2 tlc w z t =
  Re((wave_solution_voltage_phasor V1 V2 tlc w z) * cexp(ii * Cx w * t))
```

where the function `wave_solution_voltage_time` uses the phasor given by the voltage function `wave_solution_voltage_phasor` to construct the formal definition of Equation (5.25).

Next, we formally verify the general solution for voltage in the time-domain in HOL Light as follows:

Theorem 5.13. *General Solution of Wave Equation for Voltage*

```
⊢thm ∀V1 V2 R L G C w.
 let tlc = ((R,L,G,C):trans_line_const) in
 [A1] w > &0 ∧ [A2] L > &0 ∧ [A3] C > &0 ∧
 [A4] R = &0 ∧ [A5] G = &0 ∧ [A6] (∀t. Im t = &0) ∧
 [A7] (∀z. Im z = &0) ∧ [A8] Im V1 = &0 ∧ [A9] Im V2 = &0 ∧
 [A10](∀z t. V z t = Cx(wave_solution_voltage_time V1 V2 tlc w z t))
     ⇒ wave_voltage_equation V tlc z t
```

Assumptions A1-A3 ensure that the angular frequency ω, the line parameters L and C are positive real values. Assumptions A4-A5 assert that the line parameters R and G are equal to zero, which is an assumption for a lossless transmission line. Assumptions A6-A7 ensure that the imaginary parts of the variables z and t are equal to zero in the time-domain. Assumptions A8-A9 guarantee that the coefficients V1 and V2 are real. Assumption A10 provides the solution of the wave equation for voltage, i.e., Equation (5.28). The proof of the above theorem is mainly based on the following Lemma 5.1 which gives the relationship between phasor and time-domain functions as well as four important lemmas about the complex differentiation of the time-domain solution with respect to the parameters z and t, which are given in Table 5.2.

Lemma 5.1. *Relationship between Phasor and Time-Domain Functions for Voltage*

```
⊢lem ∀V1 V2 R C L G w z t.
 let tlc = ((R,L,G,C):trans_line_const) in
 [A1] w > &0 ∧ [A2] L > &0 ∧ [A3] C > &0 ∧
 [A4] R = &0 ∧ [A5] G = &0
  ⇒ wave_solution_voltage_time V1 V2 tlc w z t =
     Re(V1) * (cos(w * Re t - (Im(propagation_constant tlc w)) * Re z)) +
      Re(V2) * (cos(w * Re t + Im((propagation_constant tlc w)) * Re z))
```

Assumptions A1-A5 are the same as those of Theorem 5.13. The verification of Lemma 5.1 is mainly based on Theorem 5.8 and the properties of transcendental functions alongside some complex arithmetic reasoning.

Table 5.2: Lemmas of the Derivatives of General Solutions for Voltage in Time-Domain

Mathematical Form	Formalized Form
$\frac{\partial V(z,t)}{\partial z} = V_1 \sin(\omega t - \beta z)\beta - V_2 \sin(\omega t + \beta z)\beta$	**Lemma 1** (First-Order Partial Derivative of General Solution for Voltage with respect to distance): ∀V1 V2 R L G C w. let tlc = ((R,L,G,C):trans_line_const) [A1] w > &0 ∧ [A2] L > &0 ∧ [A3] C > &0 ∧ [A4] R = &0 [A5] G = &0 ∧ [A6] (∀t. Im t = &0) ∧ [A7] (∀z. Im z = &0) ∧ [A8] Im V1 = &0 ∧ [A9] Im V2 = &0 ∧ ⇒ complex_derivative (λz. wave_solution_voltage_time V1 V2 tlc w z t) z = Cx (Re V1 * (--sin (w * Re t - (w * sqrt (L * C)) * Re z)) (--(w * sqrt (L * C)))) + Re V2 * (--sin (w * Re t + (w * sqrt (L * C)) * Re z))* ((w * sqrt (L * C)))
$\frac{\partial^2 V(z,t)}{\partial z^2} = -V_1 \cos(\omega t - \beta z)\beta^2 - V_2 \cos(\omega t + \beta z)\beta^2$	**Lemma 2** (Second-Order Partial Derivative of General Solution for Voltage with respect to distance): ∀V1 V2 R L G C w. let tlc = ((R,L,G,C):trans_line_const) [A1] w > &0 ∧ [A2] L > &0 ∧ [A3] C > &0 ∧ [A4] R = &0 [A5] G = &0 ∧ [A6] (∀t. Im t = &0) ∧ [A7] (∀z. Im z = &0) ∧ [A8] Im V1 = &0 ∧ [A9] Im V2 = &0 ∧ ⇒ higher_complex_derivative 2 (λz. wave_solution_voltage_time V1 V2 tlc w z t) z = Cx (Re V1 * (--cos (w * Re t - (w * sqrt (L * C)) * Re z)) (w * sqrt (L * C))^2 + Re V2 * (--cos (w * Re t + (w * sqrt (L * C)) * Re z)) * ((w * sqrt (L * C))^2))
$\frac{\partial V(z,t)}{\partial t} = -V_1 \sin(\omega t - \beta z)\omega - V_2 \sin(\omega t + \beta z)\omega$	**Lemma 3** (First-Order Partial Derivative of General Solution for Voltage with respect to time): ∀V1 V2 R L G C w. let tlc = ((R,L,G,C):trans_line_const) [A1] w > &0 ∧ [A2] L > &0 ∧ [A3] C > &0 ∧ [A4] R = &0 [A5] G = &0 ∧ [A6] (∀t. Im t = &0) ∧ [A7] (∀z. Im z = &0) ∧ [A8] Im V1 = &0 ∧ [A9] Im V2 = &0 ∧ ⇒ complex_derivative (λt. wave_solution_voltage_time V1 V2 tlc w z t) t = Cx (Re V1 * (--sin (w * Re t - (w * sqrt (L * C)) * Re z)) * w + Re V2 * (--sin (w * Re t + (w * sqrt (L * C)) * Re z)) * w)
$\frac{\partial^2 V(z,t)}{\partial t^2} = -V_1 \cos(\omega t - \beta z)\omega^2 - V_2 \cos(\omega t + \beta z)\omega^2$	**Lemma 4** (Second-Order Partial Derivative of General Solution for Voltage with respect to time): ∀V1 V2 R C L G w. let tlc = ((R,L,G,C):trans_line_const) [A1] w > &0 ∧ [A2] L > &0 ∧ [A3] C > &0 ∧ [A4] R = &0 [A5] G = &0 ∧ [A6] (∀t. Im t = &0) ∧ [A7] (∀z. Im z = &0) ∧ [A8] Im V1 = &0 ∧ [A9] Im V2 = &0 ∧ ⇒ higher_complex_derivative 2 (λt. wave_solution_voltage_time V1 V2 tlc w z t) t = Cx (Re V1 * (--cos (w * Re t - (w * sqrt (L * C)) * Re z)) * w^2 + Re V2 * (--cos (w * Re t + (w * sqrt (L * C)) * Re z)) * w^2)

Similarly, we use Equation (5.20) in the time-domain solution (Equation (5.26)) for current as follows:

$$I(z,t) = \mathscr{R}e\{\frac{\gamma}{R + j\omega L}(V_0^+ e^{-\gamma z} - V_0^- e^{\gamma z})e^{j\omega t}\}$$

After rearranging the above equation, we have:

$$I(z,t) = \mathscr{R}e\{\frac{\gamma}{R + j\omega L}(V_0^+ e^{j(\omega t - \beta z)} - V_0^- e^{j(\omega t + \beta z)})\} \tag{5.29}$$

Next, by applying Euler's formula and taking the real part of the solution, we get:

$$I(z,t) = \frac{\gamma}{R + j\omega L}(V_0^+ \cos(\omega t - \beta z) - V_0^- \cos(\omega t + \beta z)) \tag{5.30}$$

Now, using Definition 5.17, we formalize the general solution (Equation (5.26)) in the time-domain for current as follows:

Definition 5.19. *General Solution for Current in Time-Domain*

```
⊢def ∀V1 V2 tlc w z t.
 wave_solution_current_time V1 V2 tlc w z t =
  Re((wave_solution_current_phasor V1 V2 tlc w z) * cexp(ii * Cx w * t))
```

where `wave_solution_current_time` accepts the phasor solution of the current `wave_solution _current_phasor` that is multiplied with the harmonic time variation term and returns its real part.

Theorem 5.14. *General Solution of Wave Equation for Current*

```
⊢thm ∀V1 V2 R L G C w.
 let tlc = ((R,L,G,C):trans_line_const) in
 [A1] w > &0 ∧ [A2] L > &0 ∧ [A3] C > &0 ∧
```

```
[A4] R = &0 ∧ [A5] G = &0 ∧ [A6] (∀t. Im t = &0) ∧

[A7] (∀z. Im z = &0) ∧ [A8] Im V1 = &0 ∧ [A9] Im V2 = &0 ∧

[A10]              Cx(&1)
      ──────────────────────────────── = &0 ∧
      characteristic_impedance tlc w

[A11] (∀z t. I z t = Cx(wave_solution_current_time V1 V2 tlc w z t))

    ⇒ wave_current_equation I tlc z t
```

Assumptions A1-A9 are the same as those of Theorem 5.13. Assumption A10 ensures that the imaginary part of the inverse characteristic impedance is equal to zero. Assumption A11 provides the solution of the wave equation for current, i.e., Equation (5.30). Similarly, the proof of Theorem 5.14 is primarily based on the formally verified lemmas about the relationship between phasor and time-domain functions, i.e., Lemma 5.2 and derivatives of the general solution for current as given in Table 5.3.

Lemma 5.2. *Relationship between Phasor and Time-Domain Functions for Current*

$\vdash_{lem}$ ∀V1 V2 R L G C w z t.

```
 let tlc = ((R,L,G,C):trans_line_const) in

 [A1] w > &0 ∧ [A2] L > &0 ∧ [A3] C > &0 ∧

 [A4] R = &0 ∧ [A5] G = &0

  ⇒ wave_solution_current_time V1 V2 tlc w z t =
                     Cx(&1)
     Re(────────────────────────────────) *
        characteristic_impedance tlc w
      (Re V1 * cos(w * Re t - Im(propagation_constant tlc w) * Re z) -

         Re V2 * cos(w * Re t + Im(propagation_constant tlc w) * Re z))
```

Assumptions A1-A5 are the same as those of Lemma 5.1. The verification of the above lemma is similar to that of Lemma 5.1.

Table 5.3: Lemmas of the Derivatives of General Solutions for Current in Time-Domain

Mathematical Form	**Formalized Form**
$\frac{\partial I(z,t)}{\partial z} = \frac{1}{Z_0}(V_1 \sin(\omega t - \beta z)\beta + V_2 \sin(\omega t + \beta z)\beta)$	**Lemma 1** (First-Order Differentiation of General Solution for Current with respect to distance): ∀V1 V2 R L G C w. let tlc = ((R,L,G,C):trans_line_const) in [A1] w > &0 ∧ [A2] L > &0 ∧ [A3] C > &0 ∧ [A4] R = &0 ∧ [A5] G = &0 ∧ [A6] (∀t. Im t = &0) ∧ [A7] (∀z. Im z = &0) ∧ [A8] Im V1 = &0 ∧ [A9] Im V2 = &0 ⇒ complex_derivative (λz. wave_solution_current_time V1 V2 tlc w z t) z = Cx (Re ($\frac{\text{Cx(\&1)}}{\text{characteristic\_impedance tlc w}}$) * (Re V1 * --sin (w * Re t - (w * sqrt (L * C)) * Re z) * --(w * sqrt (L * C)) + Re V2 * sin (w * Re t + (w * sqrt (L * C)) * Re z) * (w * sqrt (L * C))))
$\frac{\partial^2 I(z,t)}{\partial z^2} = \frac{1}{Z_0}(-V_1 \cos(\omega t - \beta z)\beta^2 + V_2 \cos(\omega t + \beta z)\beta^2)$	**Lemma 2** (Second-Order Differentiation of General Solution for Current with respect to distance): ∀V1 V2 R L G C w. let tlc = ((R,L,G,C):trans_line_const) in [A1] w > &0 ∧ [A2] L > &0 ∧ [A3] C > &0 ∧ [A4] R = &0 [A5] G = &0 ∧ [A6] (∀t. Im t = &0) ∧ [A7] (∀z. Im z = &0) ∧ [A8] Im V1 = &0 ∧ [A9] Im V2 = &0 ∧ ⇒ higher_complex_derivative 2 (λz. wave_solution_current_time V1 V2 tlc w z t) z = Cx (Re ($\frac{\text{Cx(\&1)}}{\text{characteristic\_impedance tlc w}}$) * (Re V1 * --cos (w * Re t - (w * sqrt (L * C)) * Re z) * (w * sqrt (L * C))^2 + Re V2 * cos (w * Re t + (w * sqrt (L * C)) * Re z) * (w * sqrt (L * C))^2))
$\frac{\partial I(z,t)}{\partial t} = \frac{1}{Z_0}(-V_1 \sin(\omega t - \beta z)\omega + V_2 \sin(\omega t + \beta z)\omega)$	**Lemma 3** (First-Order Differentiation of General Solution for Current with respect to time): ∀V1 V2 R L G C w. let tlc = ((R,L,G,C):trans_line_const) in [A1] w > &0 ∧ [A2] L > &0 ∧ [A3] C > &0 ∧ [A4] R = &0 [A5] G = &0 ∧ [A6] (∀t. Im t = &0) ∧ [A7] (∀z. Im z = &0) ∧ [A8] Im V1 = &0 ∧ [A9] Im V2 = &0 ∧ ⇒ complex_derivative (λt. wave_solution_current_time V1 V2 tlc w z t) t = Cx (Re ($\frac{\text{Cx(\&1)}}{\text{characteristic\_impedance tlc w}}$) * Cx (Re V1 * (--sin (w * Re t - (w * sqrt (L * C)) * Re z)) * w + Re V2 * (sin (w * Re t + (w * sqrt (L * C)) * Re z)) * w)
$\frac{\partial^2 I(z,t)}{\partial t^2} = \frac{1}{Z_0}(-V_1 \cos(\omega t - \beta z)\omega^2 + V_2 \cos(\omega t + \beta z)\omega^2)$	**Lemma 4** (Second-Order Differentiation of General Solution for Current with respect to time): ∀V1 V2 R L G C w. let tlc = ((R,L,G,C):trans_line_const) in [A1] w > &0 ∧ [A2] L > &0 ∧ [A3] C > &0 ∧ [A4] R = &0 [A5] G = &0 ∧ [A6] (∀t. Im t = &0) ∧ [A7] (∀z. Im z = &0) ∧ [A8] Im V1 = &0 ∧ [A9] Im V2 = &0 ∧ ⇒ higher_complex_derivative 2 (λt. wave_solution_current_time V1 V2 tlc w z t) t = Cx (Re ($\frac{\text{Cx(\&1)}}{\text{characteristic\_impedance tlc w}}$) * (Re V1 * --cos (w * Re t - (w * sqrt (L * C)) * Re z) * (w * sqrt (L * C))^2 + Re V2 * cos (w * Re t + (w * sqrt (L * C)) * Re z) * (w * sqrt (L * C))^2))

Since the wave and telegrapher's equations are related to each other, the general solutions of the wave equations satisfy the telegrapher's equations in the time-domain and is verified as the following HOL Light theorems:

Theorem 5.15. *General Solution of Telegrapher's Equation for Voltage*

```
⊢thm ∀V1 V2 V I R L G C w.
 let tlc = ((R,L,G,C):trans_line_const) in
 [A1] w > &0 ∧ [A2] L > &0 ∧ [A3] C > &0 ∧
 [A4] R = &0 ∧ [A5] G = &0 ∧ [A7] (∀t. Im t = &0) ∧
 [A8] (∀z. Im z = &0) ∧ [A9] Im V1 = &0 ∧ [A10] Im V2 = &0 ∧
 [A11] (∀z t. V z t = Cx(wave_solution_voltage_time V1 V2 tlc w z t))
    ⇒ telegraph_equation_voltage V I R L z t
```

Theorem 5.16. *General Solution of Telegrapher's Equation for Current*

```
⊢thm ∀V1 V2 V I R L G C w.
 let tlc = ((R,L,G,C):trans_line_const) in
 [A1] w > &0 ∧ [A2] L > &0 ∧ [A3] C > &0 ∧
 [A4] R = &0 ∧ [A5] G = &0 ∧ [A6] (∀t. Im t = &0) ∧
 [A7] (∀z. Im z = &0) ∧ [A8] Im V1 = &0 ∧ [A9] Im V2 = &0 ∧
 [A10] Im(Cx(&1) / characteristic_impedance tlc w) = &0 ∧
 [A11] (∀z t. I z t = Cx(wave_solution_current_time V1 V2 tlc w z t))
    ⇒ telegraph_equation_current V I G C z t
```

The assumptions of the above theorems are the same as those of Theorems 5.14 and 5.13. Similar to the verification of the solutions of the wave equations in the time-domain, we used Lemmas 5.1 and 5.2 as well as the verified lemmas of the derivatives for voltage and current in order to verify the correctness of the wave solutions for the telegrapher's equations. More details about the verification of the time-domain PDEs can be found in our proof script [105].

5.6 Applications

In this section, we illustrate the practical effectiveness of our proposed approach by conducting the formal analysis of the behavior of various types of transmission lines.

5.6.1 Terminated Transmission Lines

Terminated transmission lines are a type of electrical transmission line where the end is connected to a termination component, such as a resistor or an impedance-matching network. In this section, we perform a formal analysis of a terminated transmission line by formally verifying the load impedance and the voltage reflection coefficient. Moreover, we formally analyze short-circuited and open-circuited transmission lines that are commonly used in the construction of resonant circuits and matching stubs. These lines correspond to the special cases of the load impedance: $Z_L = 0$ for a short-circuited line and $Z_L = \infty$ for an open-circuited line.

Terminated transmission lines in arbitrary complex load impedances are used in the majority of sinusoidal steady-state applications. They play a vital role in ensuring a smooth transfer of signals or power, especially in applications where signal quality and system performance are critical. We consider the essential behavior of line voltage, current, and impedance for a portion of a lossless transmission line terminated with a load Z_L, as shown in Figure 5.3 [5] . In this section, we formally analyze a terminated transmission line by formally verifying in HOL Light various important properties, such as load impedance and voltage reflection coefficient.

Consider a line terminated by the load Z_L at $z = 0$ as depicted in Figure 5.3. The characteristic impedance is the ratio of the traveling voltage and current waves:

$$\frac{V_0^+}{I_0^+} = Z_0$$

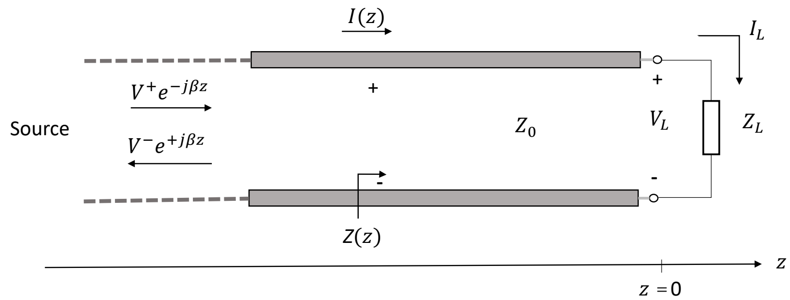


Figure 5.3: A Terminated Transmission Line

Substituting the boundary condition $z = 0$, in Equations (5.17) and (5.20), we get

$$V(0) = V_0^+ + V_0^- \tag{5.31}$$

$$I(0) = \frac{V_0^+}{Z_0} - \frac{V_0^-}{Z_0} \tag{5.32}$$

We can define the line impedance $Z(z)$ at any position z on the line as seen in Figure 5.3:

$$Z(z) = \frac{V(z)}{I(z)} = Z_0 \frac{V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z}}{V_0^+ e^{-\gamma z} - V_0^- e^{\gamma z}} \tag{5.33}$$

Here, the line impedance is not equal to Z_0 when the line is terminated, i.e., a leftward-traveling reflected wave exists. We can find the line impedance at the load position, i.e., Z_L, by dividing above two equations:

$$\frac{V(z)}{I(z)}\Big|_{z=0} = \frac{V(0)}{I(0)} = Z_L = Z_0 \frac{V_0^+ + V_0^-}{V_0^+ - V_0^-} \tag{5.34}$$

Now, we define the line impedance in HOL Light as follows:

Definition 5.20. *Line Impedance*

$\vdash_{def}$ ∀V1 V2 tlc w z.

$$\texttt{line\_impedance V1 V2 tlc w z} = \frac{\texttt{wave\_solution\_voltage\_phasor V1 V2 tlc w z}}{\texttt{wave\_solution\_current\_phasor V1 V2 tlc w z}}$$

where the HOL Light function `line_impedance` represents the ratio of the total voltage $V(z)$ to the total current $I(z)$ at any position z along the line.

Next, we formally verify that the voltage and current on the transmission line at point $z = 0$ have to abide to the boundary condition imposed by the load.

Theorem 5.17. *Line Impedance at the Load Position* ($z = 0$)

$\vdash_{thm}$ `∀V1 V2 R L G C w z.`

```
let tlc = ((R,L,G,C):trans_line_const) in
[A1] z = Cx(&0)
  ⇒ line_impedance V1 V2 tlc w z =
        characteristic_impedance tlc w * (V1 + V2)/(V1 - V2)
```

The verification of Theorem 5.17 is based on the formalizations of line and characteristic impedances alongside some complex arithmetic reasoning.

We can rearrange Equation (5.34) as the ratio of the reflected voltage amplitude to the incident voltage amplitude

$$\frac{V_0^-}{V_0^+} = \frac{Z_L - Z_0}{Z_L + Z_0} \tag{5.35}$$

This ratio of the phasors of the reverse and forward waves at the load position ($z = 0$) is defined as voltage reflection coefficient.

$$\Gamma_L = \frac{V_0^-(0)}{V_0^+(0)} = \frac{V_0^-}{V_0^+} = \frac{Z_L - Z_0}{Z_L + Z_0} \tag{5.36}$$

Next, we define the voltage reflection coefficient in HOL Light as follows:

Definition 5.21. *Voltage Reflection Coefficient*

$\vdash_{def}$ `∀V1 V2 tlc w z.`

```
voltage_reflection_coefficient V1 V2 tlc w z =
    (line_impedance V1 V2 tlc w z - characteristic_impedancetlc w) /
    (line_impedance V1 V2 tlc w z + characteristic_impedance tlc w)
```

Now, we verify that the voltage reflection coefficient is equal to the ratio of reflected voltage to the incident voltage as the following HOL Light theorem:

Theorem 5.18. *Relating Forward-Going Voltage to Reflected Voltage*

$\vdash_{thm}$ ∀V1 V2 R L G C w z.

```
let tlc = ((R,L,G,C):trans_line_const) in
[A1] V1 ≠ V2 ∧ [A2] z = Cx(&0)
[A3] characteristic_impedance tlc w ≠ Cx(&0)
  ⇒ voltage_reflection_coefficient V1 V2 tlc w z = V2 / V1
```

Assumption A1 ensures that voltages are different from each other. Assumption A2 represents the boundary condition $z = 0$. Assumption A3 guarantees that the characteristic impedance is nonzero. The verification of the above theorem is mainly based on Theorem 5.17 along with some complex arithmetic reasoning.

We can also obtain the line impedance at the load ($z = 0$) from the reflection coefficient by rewriting the relationship in Equation (5.36):

$$Z_L = Z_0 \frac{1 + \Gamma_L}{1 - \Gamma_L} \tag{5.37}$$

Here, the quantity Γ_L is known as the voltage reflection coefficient. Now, we verify the above relationship as the following HOL Light theorem.

Theorem 5.19. *Final Equation for Line Impedance at the Load Position*

$\vdash_{thm}$ ∀V1 V2 R L G C w z.

```
let tlc = ((R,L,G,C):trans_line_const) in
[A1] V1 ≠ V2 ∧ [A2] z = Cx(&0) ∧
[A3] characteristic_impedance tlc w ≠ Cx(&0)
 ⇒ line_impedance V1 V2 tlc w z = characteristic_impedance tlc w *
     (Cx(&1) + voltage_reflection_coefficient V1 V2 tlc w z) /
     (Cx(&1) - voltage_reflection_coefficient V1 V2 tlc w z)
```

Assumptions `A1-A3` are the same as those of Theorem 5.18. The verification of the above theorem is primarily based on Theorems 5.17 and 5.18 alongside some complex arithmetic reasoning.

In the following subsections, we formally analyze short-circuited and open-circuited transmission lines as special cases of a terminated transmission line.

5.6.1.1 Short-Circuited Line

When the load end of a transmission line is connected in such a way that it creates a short circuit, it is referred to as a short-circuited transmission line. These lines are extensively used in microwave engineering and Radio-Frequency (RF) systems to ensure a proper impedance matching, which is essential for an efficient power transmission and preserving the integrity of signals. Figure 5.4 [5] depicts a transmission line of length l that is terminated by a short circuit ensuring a zero load impedance, i.e., $Z_L = 0$.

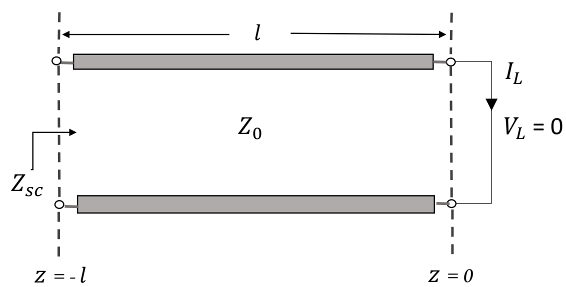


Figure 5.4: Short-Circuited Line

Moreover, the short-circuited termination forces the load voltage V_L to zero. Therefore, from Equation (5.17), we have:

$$V_L = V(z)|_{z=0} = 0$$

$$V^+e^{-j\beta z} + V^-e^{j\beta z}|_{z=0} = 0$$

$$V^+ + V^- = 0$$

This implies

$$V^- = -V^+ \tag{5.38}$$

We employ Equation (5.20) to find the load current flowing through the short circuit by utilizing Equation (5.38) as:

$$I_L = I(z)|_{z=0}$$

$$= \frac{1}{Z_0}[V^+ - V^-|_{z=0}$$

$$= \frac{2V^+}{Z_0} \tag{5.39}$$

Everywhere else on the transmission line, the voltage and current are mathematically expressed as [5]:

$$V(z) = V^+(e^{-j\beta z} - e^{j\beta z}) = -2V^+ j\sin(\beta z)$$

$$I(z) = \frac{V^+}{Z_0}(e^{-j\beta z} + e^{j\beta z}) = \frac{2V^+}{Z_0}\cos(\beta z)$$

The line impedance observed when looking towards the far end (short-circuited location) on the transmission line is:

$$Z(z) = \frac{V(z)}{I(z)} = Z_0 \frac{-2V^+ j \sin(\beta z)}{2V^+ \cos(\beta z)} = -jZ_0 \tan(\beta z)$$

Next, we formally verify the short-circuited line in HOL Light as follows:

Theorem 5.20. *Short-Circuited Line*

```
⊢thm ∀V1 V2 R L G C w z.
 let tlc = ((R,L,G,C):trans_line_const) in
 [A1] (V2 = --V1) ∧ [A2] w > &0 ∧ [A3] L > &0 ∧
 [A4] C > &0 ∧ [A5] R = &0 ∧ [A6] G = &0 ∧ [A7] V1 ≠ Cx (&0)
   ⇒ line_impedance V1 V2 tlc w z =
         --ii * characteristic_impedance tlc w *
            ctan(Cx(Im(propagation_constant tlc w)) * z)
```

Assumption A1 provides the condition for the short-circuited line. Assumptions A2-A4 guarantee that the angular frequency ω and the parameters L and C cannot be negative or zero, respectively. Assumptions A5-A6 assert that the line parameters R and G are equal to zero, which are assumptions for a lossless transmission line. Assumption A7 provides that the coefficient V1 is different than zero. The verification of Theorem 5.20 is primarily based on the following lemma:

Lemma 5.3. *Lemma for Short-Circuited Line*

```
⊢lem ∀V1 V2 R L G C w z.
 let tlc = ((R,L,G,C):trans_line_const) in
 [A1] V2 = --V1 ∧ [A2] w > &0 ∧ [A3] L > &0 ∧
 [A4] C > &0 ∧ [A5] R = &0 ∧ [A6] G = &0 ∧ [A7] V1 ≠ Cx(&0)
  ⇒ line_impedance V1 V2 tlc w z = characteristic_impedance tlc w *
```

$$\frac{\texttt{--Cx(\&2)} * \texttt{ii} * \texttt{V1} * \texttt{csin(Cx(Im(propagation\_constant tlc w))} * \texttt{z)}}{\texttt{Cx(\&2)} * \texttt{V1} * \texttt{ccos(Cx(Im(propagation\_constant tlc w))} * \texttt{z)}}$$

Every assumption in the above lemma is the same as that of Theorem 5.20. The proof of Lemma 5.3 is mainly based on Theorems 5.8 and 5.9, properties of the trancendental functions along with some complex arithmetic reasoning.

5.6.1.2 Open-Circuited Line

When a transmission line is open at the load end, it is referred to as an open-circuited transmission line. Since the terminal is characterized by an open circuit configuration, the signal or current is unable to propagate beyond the open-circuited point. Open-circuited transmission lines are employed in antenna design to model the behavior of open-ended radiating devices. Figure 5.5 [5] depicts an open-circuited transmission line with an infinite load impedance, i.e., $Z_L = \infty$.

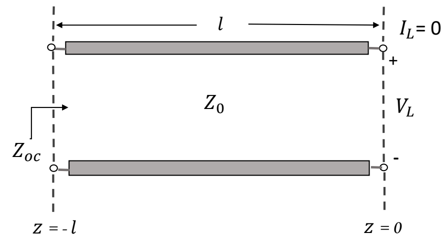


Figure 5.5: Open-Circuited Line

An open-circuited transmission line forces the load current I_L to be zero. Therefore, by using Equation (5.20) we have:

$$I_L = I(z)|_{z=0} = 0$$

$$\frac{V^+}{Z_0}e^{-j\beta z} - \frac{V^-}{Z_0}e^{j\beta z}|_{z=0} = 0$$

$$\frac{V^+ + V^-}{Z_0} = 0$$

Thus,

$$V^- = V^+ \tag{5.40}$$

Note that the load voltage V_L appearing across the open circuit can be found from Equation (5.17) using Equation (5.40):

$$V_L = V(z)|_{z=0}$$

$$= V^+ e^{-j\beta z} + V^- e^{j\beta z}|_{z=0}$$

$$= V^+ + V^- = 2V^+ \tag{5.41}$$

Everywhere else on the transmission line, the voltage and current are mathematically expressed as [5]:

$$V(z) = V^+(e^{-j\beta z} + e^{j\beta z}) = 2V^+ \cos(\beta z)$$

$$I(z) = \frac{V^+}{Z_0}(e^{-j\beta z} - e^{j\beta z}) = -\frac{2V^+}{Z_0} j \sin(\beta z) = \frac{2V^+}{Z_0} e^{-j\pi/2} \sin(\beta z)$$

Next, we formally verify the open-circuited line in HOL Light as follows:

Theorem 5.21. *Open-Circuited Line*

```
⊢thm ∀V1 V2 R L G C w z.
  let tlc = ((R,L,G,C):trans_line_const) in
  [A1] (V2 = V1) ∧ [A2] w > &0 ∧ [A3] L > &0 ∧
  [A4] C > &0 ∧ [A5] R = &0 ∧ [A6] G = &0 ∧ [A7] V1 ≠ Cx(&0)
    ⇒ line_impedance V1 V2 tlc w z = ii * characteristic_impedance tlc w *
                                 ccot (Cx(Im(propagation_constant tlc w)) * z)
```

Assumption A1 ensures the condition for the open-circuited line. The rest of the assumptions are the same as that of Theorem 5.20.

Similar to Theorem 5.20, the proof of the above theorem is mainly based on the following lemma:

Lemma 5.4. *Lemma for Open-Circuited Line*

$\vdash_{lem}$ ∀V1 V2 R L G C w z.

let tlc = ((R,L,G,C):trans_line_const) in

[A1] (V2 = V1) ∧ [A2] w > &0 ∧ [A3] L > &0 ∧

[A4] C > &0 ∧ [A5] R = &0 ∧ [A6] G = &0 ∧ [A7] V1 ≠ Cx (&0) ∧

$$\Rightarrow \texttt{line\_impedance V1 V2 tlc w z = characteristic\_impedance tlc w} * \frac{\texttt{Cx(\&2)} * \texttt{ii} * \texttt{V1} * \texttt{ccos(Cx(Im(propagation\_constant tlc w))} * \texttt{z)}}{\texttt{--Cx(\&2)} * \texttt{V1} * \texttt{csin(Cx(Im(propagation\_constant tlc w))} * \texttt{z)}}$$

The proof of the above lemma is mainly based on the formally verified lemmas about the exponential functions alongwith some complex arithmetic reasoning. This completes the formal analysis of the terminated, short-circuited and open-circuited transmission lines. The details about the analysis can be found in the related proof script [105].

5.6.2 Coupled Transmission Lines

In various transmission line applications, the proximity of neighboring lines often results in a level of coupling. This close proximity leads to modifications in the electromagnetic fields, consequently influencing the propagating voltage and current waves and in turn, altering the characteristic impedance of the transmission line. While this coupling may pose a drawback where it leads to undesired signals, commonly referred to as *cross-talk*, it can also serve as a mean of intentionally transferring a set amount of signal to another circuit for various purposes such as monitoring, measurement, or signal processing [106]. There exist two forms of coupling, namely electric and magnetic. The electric coupling results from charges on one line inducing charges on another, often explained by mutual capacitance. The magnetic coupling, on the other hand, arises from the interaction of magnetic flux between the lines and is typically described by mutual inductance. Figure 5.6 [107] shows a generic circuit model for Coupled Transmission Lines (CTLs). Under the assumption of lossless conditions, we consider two isolated transmission lines characterized by distributed inductances and capacitances per unit length, represented as L_i and C_i for $i = 1, 2$. The respective propagation velocities and characteristic impedances are defined as $v_i = 1/\sqrt{L_i C_i}$ and $Z_i = \sqrt{L_i/C_i}$, respectively. To model the interaction between these lines, mutual inductance and capacitance per unit length, denoted as L_m and C_m, are introduced.

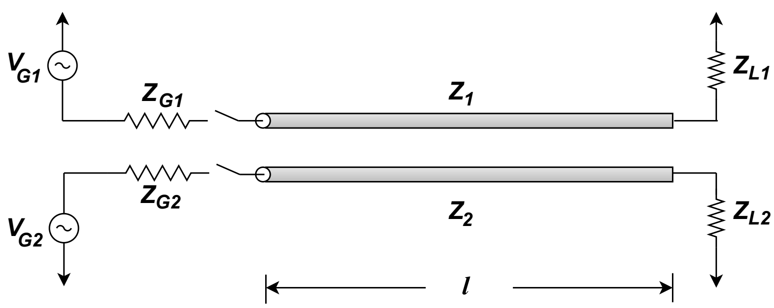


Figure 5.6: Coupled Transmission Lines

The dynamics of the CTLs can then be mathematically described as follows [106]:

$$\frac{\partial V_1}{\partial z} = -L_1 \frac{\partial I_1}{\partial t} - L_m \frac{\partial I_2}{\partial t} \tag{5.42}$$

$$\frac{\partial V_2}{\partial z} = -L_2 \frac{\partial I_2}{\partial t} - L_m \frac{\partial I_1}{\partial t} \tag{5.43}$$

$$\frac{\partial I_1}{\partial z} = -C_1 \frac{\partial V_1}{\partial t} + C_m \frac{\partial V_2}{\partial t} \tag{5.44}$$

$$\frac{\partial I_2}{\partial z} = -C_2 \frac{\partial V_2}{\partial t} + C_m \frac{\partial V_1}{\partial t} \tag{5.45}$$

These equations are generalizations of the telegrapher's equations incorporating the mutual inductance and capacitance, which were originally developed for a single transmission line.

To overcome the considerable challenges of solving time-domain PDEs, we utilize the *phasor* concept to transform them into a set of coupled Ordinary Differential Equations (ODEs) for the voltages and currents. For sinusoidal steady-state (*phasor*) excitation of the lines, we obtain by replacing $\partial/\partial t \Rightarrow j\omega$ [108]:

$$\frac{dV_1}{dz} = -j\omega L_1 I_1(z) - j\omega L_m I_2(z) \tag{5.46}$$

$$\frac{dV_2}{dz} = -j\omega L_m I_1(z) - j\omega L_2 I_2(z) \tag{5.47}$$

$$\frac{dI_1}{dz} = -j\omega C_1 V_1(z) + j\omega C_m V_2(z) \tag{5.48}$$

$$\frac{dI_2}{dz} = j\omega C_m V_1(z) - j\omega C_2 V_2(z) \tag{5.49}$$

In the next subsections, we briefly present the formal modeling and verification of coupled transmission lines using the generalized telegrapher's equations. More details regarding the formalization of this application and the verification of the solutions of the related equations are available in [109].

5.6.2.1 Formal Modeling of Coupled Transmission Lines

We first formalize Equations (5.46) and (5.47) capturing the voltages on CTLs in HOL Light as follows:

Definition 5.22. *First Equation for Voltage*

```
⊢def ∀V1 V2 I1 I2 L1 L2 Lm w z.
  coupled_vol_ode_fst ((V1,V2),(I1,I2))(L1,L2),Lm) z ⇔
    complex_derivative (λz. V1(z)) z =
        --ii * Cx w * (Cx L1 * I1(z) + Cx Lm * I2(z))
```

Definition 5.23. *Second Equation for Voltage*

```
⊢def ∀V1 V2 I1 I2 L1 L2 Lm w z.
  coupled_vol_ode_snd ((V1,V2),(I1,I2))(L1,L2),Lm) z  ⇔
    complex_derivative (λz. V2(z)) z =
        --ii * Cx w * (Cx Lm * I1(z) + Cx L2 * I2(z))
```

where `coupled_vol_ode_fst` and `coupled_vol_ode_snd` use the complex-derivative function in HOL Light to model the telegrapher's equations.
The variables `L1`:ℝ and `Lm`:ℝ represent the distributed and mutual inductance per unit length, respectively. Here, the variables `z`:ℂ, and `w`:ℝ denote the spatial coordinate and the angular frequency, respectively.

Similarly, we can formalize Equations (5.48) and (5.49) capturing the currents on CTLs as:

Definition 5.24. *First Equation for Current*

```
⊢def ∀V1 V2 I1 I2 C1 C2 Cm w z.
  coupled_cur_ode_fst ((V1,V2),(I1,I2))(C1,C2),Cm) z ⇔
```

```
complex_derivative (λz.  I1(z)) z =
    --ii * Cx w * (Cx (C1) * V1(z) - Cx (Cm) * V2(z))
```

Definition 5.25. *Second Equation for Current*

```
⊢def ∀V1 V2 I1 I2 C1 C2 Cm w z.
  coupled_cur_ode_snd ((V1,V2),(I1,I2))(C1,C2),Cm) z  ⇔
    complex_derivative (λz.  I2(z)) z =
      --ii * Cx w * (--Cx (Cm) * V1(z) + Cx (C2) * V2(z))
```

5.6.2.2 Formal Verification of Coupled Transmission Lines

To simplify the analysis of the telegrapher's equations, we consider the scenario of the identical transmission lines. In this case, we have $L_1 = L_2 \equiv L_0$ and $C_1 = C_2 \equiv C_0$, so that $\beta_1 = \beta_2 = \omega\sqrt{L_0 C_0} \equiv \beta$ and $Z_1 = Z_2 = \sqrt{L_0/C_0} \equiv Z_0$. Additionally, the wave propagation speed is defined as $v_0 = 1/\sqrt{L_0 C_0}$. If two lossless coupled lines have the same self-inductance parameters $L_1 = L_2 \equiv L_0$ and self-capacitance parameters $C_1 = C_2 \equiv C_0$, the coupled-line structure is considered symmetric. The final solution for symmetric coupled lines can be efficiently derived by combining two single-line scenarios. This is achieved by applying two specific types of excitations: *even* and *odd* modes. In the *even* mode, currents in the conductors exhibit equal magnitudes and flow in parallel directions, while in the *odd* mode, currents in the conductors possess equal magnitudes but flow in opposite directions.

We now mathematically express the solutions of the telegrapher's equations for the CTLs in terms of even and odd modes for the voltages and currents as follows:

$$V_1(z) = \underbrace{\frac{e^{-j\beta_+ z} + \Gamma_{L+} e^{-2j\beta_+ l} e^{j\beta_+ z}}{1 - \Gamma_{G+}\Gamma_{L+} e^{-2j\beta_+ l}} V_+}_{\text{even}} + \underbrace{\frac{e^{-j\beta_- z} + \Gamma_{L-} e^{-2j\beta_- l} e^{j\beta_- z}}{1 - \Gamma_{G-}\Gamma_{L-} e^{-2j\beta_- l}} V_-}_{\text{odd}} \tag{5.50}$$

$$V_2(z) = \underbrace{\frac{e^{-j\beta_+ z} + \Gamma_{L+} e^{-2j\beta_+ l} e^{j\beta_+ z}}{1 - \Gamma_{G+}\Gamma_{L+} e^{-2j\beta_+ l}} V_+}_{\text{even}} - \underbrace{\frac{e^{-j\beta_- z} + \Gamma_{L-} e^{-2j\beta_- l} e^{j\beta_- z}}{1 - \Gamma_{G-}\Gamma_{L-} e^{-2j\beta_- l}} V_-}_{\text{odd}} \tag{5.51}$$

Similarly, the general solutions for the currents can be mathematically express as:

$$I_1(z) = \frac{1}{Z_+} \left[\underbrace{\frac{e^{-j\beta_+ z} - \Gamma_{L+} e^{-2j\beta_+ l} e^{j\beta_+ z}}{1 - \Gamma_{G+}\Gamma_{L+} e^{-2j\beta_+ l}} V_+}_{\text{even}} + \underbrace{\frac{e^{-j\beta_- z} - \Gamma_{L-} e^{-2j\beta_- l} e^{j\beta_- z}}{1 - \Gamma_{G-}\Gamma_{L-} e^{-2j\beta_- l}} V_-}_{\text{odd}} \right] \tag{5.52}$$

$$I_2(z) = \frac{1}{Z_-} \left[\underbrace{\frac{e^{-j\beta_+ z} - \Gamma_{L+} e^{-2j\beta_+ l} e^{j\beta_+ z}}{1 - \Gamma_{G+}\Gamma_{L+} e^{-2j\beta_+ l}} V_+}_{\text{even}} - \underbrace{\frac{e^{-j\beta_- z} - \Gamma_{L-} e^{-2j\beta_- l} e^{j\beta_- z}}{1 - \Gamma_{G-}\Gamma_{L-} e^{-2j\beta_- l}} V_-}_{\text{odd}} \right] \tag{5.53}$$

In this context, the parameters $\beta_\pm$ and $Z_\pm$ indicate the wave numbers and the impedances, respectively and they can be mathematically express as follows:

$$\begin{aligned} \beta_+ &= \omega\sqrt{(L_0 + L_m)(C_0) - C_m} \\ \beta_- &= \omega\sqrt{(L_0 - L_m)(C_0) + C_m} \end{aligned} \tag{5.54}$$

and

$$\begin{aligned} Z_+ &= \sqrt{\frac{L_0 + L_m}{C_0 - C_m}} \\ Z_- &= \sqrt{\frac{L_0 - L_m}{C_0 + C_m}} \end{aligned} \tag{5.55}$$

To formalize the general solutions of the telegrapher's equations for voltages and currents, we first define the reflection coefficients `g1`, `g2`, `g3`, and `g4`, corresponding to Γ_{L+}, Γ_{G+}, Γ_{L-}, and Γ_{G-}. We also define the transmission line parameters `L1`, `L2`, C1, and C2 for identical lines as 4-tuples, along with the complex constants Vp and Vm, associated with V_+ and V_-, in HOL Light.

```
new_type_abbrev (``ref_coef'',`:  (g1 × g2 × g3 × g4)`)

new_type_abbrev (``ind_cap'',`:  (L1 × L2 × C1 × C2)`)

new_type_abbrev (``vol_cons'',`:  (Vp × Vm )`)
```

In addition, the types of coefficients are given in Table 5.4.

Table 5.4: Data Types of Coefficients

Parameter Description	Standard Symbol	HOL Light Symbol: Type
Reflection coefficient at the load in even mode	Γ_{L+}	g1: $\mathbb{C}$
Reflection coefficient at the generator in even mode	Γ_{G+}	g2: $\mathbb{C}$
Reflection coefficient at the load in odd mode	Γ_{L-}	g3: $\mathbb{C}$
Reflection coefficient at the generator in odd mode	Γ_{G-}	g4: $\mathbb{C}$
Complex constant	V_+	Vm: $\mathbb{C}$
Complex constant	V_-	Vp: $\mathbb{C}$

We now present the formalization of the general solutions of the telegrapher's equations (Equations (5.46)-(5.49)) for voltage and current in vector form. For brevity, we only provide the solutions for the first voltage and current, i.e., Equations (5.50) and (5.52). These solutions are formalized in HOL Light as follows:

Definition 5.26. *Vector Form of the General Solutions for the Voltages*

```
⊢def ∀Vm Vp V1 V2 L0 Lm C0 Cm I1 I2 g1 g2 g3 g4 z l w.
vol_sol_vec ((V1,V2),(I1,I2))(Vm,Vp)((L0,Lm),(C0,Cm))(g1,g2,g3,g4) z l w ⇔
  (let vlcr = ((V1,V2),(I1,I2)) and tlc = ((L0,Lm),(C0,Cm)) and
        rc = (g1,g2,g3,g4) and vc = (Vm,Vp) in
```

```
        vector[V1 z; V2 z] = vector[vol_sol_fst vc tlc rc z l w;
                                    vol_sol_snd vc tlc rc z l w])
```

Here, vol_sol_fst and vol_sol_snd represent the formalization of the general solutions for the voltages.

Definition 5.27. *Vector Form of the General Solutions for the Currents*

```
⊢def ∀Vm Vp V1 V2 L0 Lm C0 Cm I1 I2 g1 g2 g3 g4 z l w.
vol_sol_vec ((V1,V2),(I1,I2))(Vm,Vp)((L0,Lm),(C0,Cm))(g1,g2,g3,g4) z l w ⇔
  (let vlcr = ((V1,V2),(I1,I2)) and tlc = ((L0,Lm),(C0,Cm)) and
        rc = (g1,g2,g3,g4) and vc = (Vm,Vp) in
        vector[I1 z; I2 z] = vector[cur_sol_fst vc tlc rc z l w;
                                    cur_sol_snd vc tlc rc z l w])
```

Similarly, cur_sol_fst and cur_sol_snd represent the formalization of the general solutions for the currents. The final step is to formally verify the correctness of the solutions of the generalized telegrapher's equations as the following HOL Light theorem:

Theorem 5.22. *Verification of the General Solutions of the Telegrapher's Equation*

```
⊢thm ∀V1 V2 I1 I2 C1 C2 L1 L2 Vm Vp L0 Lm C0 Cm g1 g2 g3 g4 l w.
   let tlc = ((L0,Lm),(C0,Cm)) and ind = ((L1,L2),Lm) and
       cap = (C1,C2),Cm) and vlcr = ((V1,V2),(I1,I2)) and
        rc = (g1,g2,g3,g4) and vc = (Vm,Vp) in
    [A1] &0 < L1 ∧ [A2] &0 < L2 ∧ [A3] &0 < C1 ∧ [A4] &0 < C2
    [A5] Cm < C0 ∧ [A6] Lm < L0 ∧ [A7] &0 < Cm ∧ [A8] &0 < Lm
    [A9] L1 = L0 ∧ [A10] L2 = L0 ∧ [A11] C1 = C0 ∧ [A12] C2 = C0
    [A13] (∀z.  vol_sol_vec vlcr tlc rc z l w) ∧
    [A14] (∀z.  cur_sol_vec vlcr tlc rc z l w)
    ⇒ vol_ode_mat_rep vlcr ind w z ∧ cur_ode_mat_rep vlcr cap w z
```

Assumptions `A1-A4` ensure that the inductances and capacitances are positive quantities. Assumptions `A5-A6` indicate that the distributed capacitance and inductance are greater than the mutual inductance and capacitance, respectively. Assumptions `A7-A8` guarantee that the mutual capacitance and inductance are greater than zero. Assumptions `A9-A12` model the conditions pertaining identical transmission lines. Assumptions `A13` and `A14` provide the general solutions of the telegrapher's equations for the voltages and the currents in vector form. Finally, the conclusion of the theorem presents the generalized telegrapher's equations, i.e., Equations (5.46)-(5.49). The verification of Theorem 5.22 is mainly based on the following four important formally verified lemmas about the complex derivatives of the general solutions. Details about these lemmas and their proofs can be found in the related HOL Light script [110].

5.6.3 Cascaded Transmission Lines

The applications presented in the previous sections regarding terminated and coupled transmission lines may reveal some limitations such as impedance mismatch and crosstalk, which complicates their design and analysis. In general, transmission lines are categorized as short, medium or long transmission lines. For instance, long transmission lines are essential for effective power and data transmission over long distances. Furthermore, to represent some scenarios, like impedance matching and operational flexibility, a single classification may not be adequate, necessitating the use of cascaded transmission lines, where these lines are connected in series, such as short-short or short-medium and vice versa [108]. These lines are generally represented as two-port networks that are electrical circuits comprising of lumped elements, such as resistors, capacitors and inductors, with pairs of terminals on sending and receiving ends, enabling a connection to the external networks [111]. Such two-port networks are traditionally modeled using, so-called, *ABCD parameters* [112], which are closely related to the telegrapher's equations as both are used to describe the behavior of transmission lines in electrical networks. For example, long transmission lines are modeled using the telegrapher's equations to account for distributed electrical properties, and their behavior is compactly represented by ABCD parameters. This representation provide a more comprehensive and versatile framework that can accommodate various transmission lines configurations, particularly in complex, multi-line, or long-distance systems. After a brief overview of ABCD parameters based models for short, medium transmission and cascaded transmission lines, in the rest of this section, we present the formal modeling and analysis of cascaded transmission lines in HOL Light. For the sake of conciseness, we omitted the details of the analysis of these transmission lines where more details can be found in [113].

5.6.3.1 ABCD Parameters of a Transmission Line

ABCD parameters enable developing the mathematical models (system of equations) of transmission networks capturing a relationship between the sending and receiving quantities, i.e., voltages and currents, by incorporating their characteristics like impedance and admittance. Moreover, these mathematical models can also be derived from the application of the physical laws, such as the Kirchhoff's Current Law (KCL) and the Kirchhoff's Voltage Law (KVL) on the two-port networks corresponding to the transmission lines. Figure 5.7 illustrates a generic two-port network, which is to be characterized using ABCD parameters.

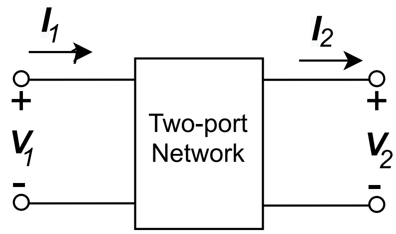


Figure 5.7: Two-port Network

In the context of the two-port transmission line model, port 1 is characterized by an input current, denoted as I_1, and a corresponding input voltage, V_1. The resulting output voltage and current at port 2 are labeled as V_2 and I_2, respectively. It is essential to note that the chosen current directions designate I_1 as entering and I_2 as leaving the two-port network. Here, V_1 and I_1 are dependent variables whereas V_2 and I_2 are considered independent. Let A, B, C and D be constants that characterize the above network. These ABCD parameters relate the input variables V_1 and I_1 as functions of the output variables V_2 and I_2 as follows:

$$V_1 = AV_2 + BI_2$$

$$I_1 = CV_2 + DI_2$$

The above equations can be written in matrix form:

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix}$$

These parameters play an important role in the analysis and comprehension of the transmission of electrical signals through intricate networks. It is also utilized to evaluate the performance of input, output voltage, and current within the transmission network.

Now, we model the ABCD parameters and transmission line constants as 4-tuples in HOL Light as:

```
new_type_abbrev (``abcd_param'',`:(A × B × C × D)')
new_type_abbrev (``trans_lines_const'',`:(R × L × Ca × G)')
```

Similarly, we model the types of the sending and receiving end quantities as follows:

```
new_type_abbrev (``send_end_quan'',`:(Vs × Is)')
new_type_abbrev (``receive_end_quan'',`:(VR × IR)')
```

where the types of A, B, C, D, R, L, Ca, G, Vs, Is, VR and IR are given in Table 5.5.

Table 5.5: Parameters, and their Standard and HOL Light Symbols

Parameter Description	Standard Symbol	HOL Light Symbol:Type	Parameter Description	Standard Symbol	HOL Light Symbol:Type
Parameter A	A	A:$\mathbb{R}$	Parameter B	B	B:$\mathbb{R}$
Parameter C	C	C:$\mathbb{R}$	Parameter D	D	D:$\mathbb{R}$
Resistance	R	R:$\mathbb{R}$	Inductance	L	L:$\mathbb{R}$
Capacitance	Ca	Ca:$\mathbb{R}$	Conductance	G	G:$\mathbb{R}$
Sending End Voltage	V_s	Vs:$\mathbb{C} \to \mathbb{C}$	Sending End Current	I_s	Is:$\mathbb{C} \to \mathbb{C}$
Receiving End Voltage	V_R	VR:$\mathbb{C} \to \mathbb{C}$	Receiving End Current	I_R	IR:$\mathbb{C} \to \mathbb{C}$

In the following subsection, we briefly present the formalization of transmission lines in the HOL Light proof assistant.

5.6.3.2 Formalization of the Transmission Lines Models

Now, we present the formalization of the ABCD matrices for the lumped circuits models such as short and medium transmission lines. In our formalization, we present these transmission lines models, based on their type, as an enumerated type definition in HOL Light as:

```
define_type ‘‘tl_models = ShortTL_SerImp | ShortTL_ShuAdm |
                          MediumTL_TCir | MediumTL_PiCir’’
```

A transmission line is valid if it is represented by a valid ABCD parameters based model and satisfies various constraints on transmission line constants.
We formalize the validity of a transmission line in HOL Light as follows:

Definition 5.28. *Valid Transmission Line*

```
⊢def ∀tlc tlm.  valid_transmission_line (tlm,tlc) ⇔
                  valid_transm_line_model tlm ∧ valid_tl_const tlc
```

where the function `valid_transm_line_model` accepts a transmission line model, i.e., short or medium and provides its validity. This is formalized as:

Definition 5.29. *Valid Transmission Line Models*

```
⊢def  (valid_transm_line_model (ShortTL_SerImp) ⇔ T) ∧
      (valid_transm_line_model (ShortTL_ShuAdm) ⇔ T) ∧
      (valid_transm_line_model (MediumTL_TCir) ⇔ T) ∧
      (valid_transm_line_model (MediumTL_PiCir) ⇔ T)
```

Similarly, the function `valid_tl_const` in Definition 5.28 models the validity of the transmission line constants as follows:

Definition 5.30. *Valid Transmission Line Constants*

```
⊢def ∀R L Ca G. valid_tl_const ((R,L,Ca,G):trans_lines_const) ⇔
            &0 < R ∧ &0 < L ∧ &0 < Ca ∧ &0 < G
```

The verification of a relationship between the sending and receiving end quantities (voltages and currents) ensures the correct working of the ABCD parameters based models of transmission lines. To verify this relationship, we first model the generalized ABCD matrix as follows:

Definition 5.31. *Generalized ABCD Matrix*

$\vdash_{def}$ ∀A B C D. abcd_mat_gen ((A,B,C,D):abcd_param) = $\begin{bmatrix} \texttt{A} & \texttt{B} \\ \texttt{C} & \texttt{D} \end{bmatrix}$

Next, we model the sending and receiving quantities as two-dimensional vectors in HOL Light:

Definition 5.32. *Sending End Vector*

$\vdash_{def}$ ∀Vs Is z. send_end_vec ((Vs,Is):send_end_quan) z = $\begin{bmatrix} \texttt{Vs(z)} \\ \texttt{Is(z)} \end{bmatrix}$

Definition 5.33. *Receiving End Vector*

$\vdash_{def}$ ∀VR IR z. recei_end_vec ((Vs,Is):recei_end_quan) z = $\begin{bmatrix} \texttt{VR(z)} \\ \texttt{IR(z)} \end{bmatrix}$

Now, we use Definitions 5.31, 5.32 and 5.33 to formalize a generalized relationship between the sending and receiving quantities of transmission lines as:

Definition 5.34. *Relationship Between Sending/Receiving Quantities*

$\vdash_{def}$ ∀Vs Is VR IR A B C D.

relat_send_receive_quan_gen (Vs,Is) (VR,IR) (A,B,C,D) z ⇔

(send_end_vec (Vs,Is) z = ((abcd_mat_gen (A,B,C,D)):comp_mat) **

recei_end_vec (VR,IR) z)

Finally, we formalize KCL and KVL, capturing the dynamics of voltage and current in the circuit models, as follows:

Definition 5.35. *KCL and KVL*

$\vdash_{def}$ ∀cur_list z. kcl (cur_list:cur_fun list) (z:complex) =

((vsum (0..(LENGTH (cur_list) - 1)) (λn. EL n cur_list z)) = Cx (&0))

$\vdash_{def}$ ∀vol_list z. kvl (vol_list:vol_fun list) (z:complex) =

((vsum (0..(LENGTH (vol_list) - 1)) (λn. EL n vol_list z)) = Cx (&0))

where the functions kcl and kvl accept lists of currents and voltages across the components of circuits, cur_list and vol_list, a complex variable z, and return the implementations of KCL and KVL, respectively. For example, kcl ensures that the sum of all currents leaving a particular node is zero. Here, the function vsum accepts a vector-valued function f and provides the summation $\sum_{i=0}^{n} f(i)$. Similarly, the function EL n l extracts the n^{th} element from a list l.

Next, we formalize the ABCD matrices for the lumped circuits models, namely, short and medium transmission lines in HOL Light as follows:

Definition 5.36. *ABCD Matrices of Transmission Lines Models*

$\vdash_{def}$ ∀R L Ca G w.

abcd_mat ShortTL_SerImp ((R,L,Ca,G):trans_lines_const) w =

$$\begin{bmatrix} 1 & \mathsf{R} + \mathsf{ii} * \mathsf{w} * \mathsf{L} \\ 0 & 1 \end{bmatrix} \quad \wedge$$

abcd_mat ShortTL_ShuAdm ((R,L,Ca,G):trans_lines_const) w =

$$\begin{bmatrix} 1 & 0 \\ \frac{1}{\mathsf{R}} + \mathsf{ii} * \mathsf{w} * \mathsf{Ca} & 1 \end{bmatrix} \quad \wedge$$

abcd_mat MediumTL_PiCir ((R,L,Ca,G):trans_lines_const) w =

$$\begin{bmatrix} 1 + \frac{(\mathsf{w} * \mathsf{Ca})(\mathsf{R} + \mathsf{ii} * \mathsf{w} * \mathsf{L})}{2} & 0 \\ \frac{1}{\mathsf{R}} + \mathsf{ii} * \mathsf{w} * \mathsf{Ca} & 1 \end{bmatrix} \quad \wedge$$

abcd_mat MediumTL_TCir ((R,L,Ca,G):trans_lines_const) w =

$$\begin{bmatrix} \mathsf{T}\left(1 + \frac{(\mathsf{w} * \mathsf{Ca})\mathsf{T}}{4}\right) & \mathsf{w} * \mathsf{Ca} \\ \left(\frac{1}{\mathsf{R}}\right) + \mathsf{ii} * \mathsf{w} * \mathsf{Ca} & \left(1 + \frac{(\mathsf{w} * \mathsf{Ca})\mathsf{T}}{2}\right) \end{bmatrix}$$

where T = R + ii ∗ w ∗ L. The function abcd_mat accepts the model, transmission line constants and a variable w, and returns the ABCD matrix corresponding to the given model. Here, the symbol "ii" is used to represent the imaginary part.

It is important to note that single transmission line models can be categorized based on the arrangement of various lumped components of the circuits. For example, a short circuit representation, where a resistor R is connected in series to a capacitor Ca is commonly known as a *series impedance model* for the short transmission line. Similarly, medium transmission lines can be represented by their Π or T models based on the connection of various circuit components [111]. Therefore, it is sufficient to model single transmission lines that can further be used in the cascaded transmission lines, connected in series.

5.6.3.3 Formalization of Cascaded Transmission Lines

When multiple transmission lines models are connected in series, it results into a cascaded transmission line as depicted in Figure 5.8. Here, n transmission lines are connected in series and a relationship between the sending and receiving end quantities is mathematically expressed as follows:

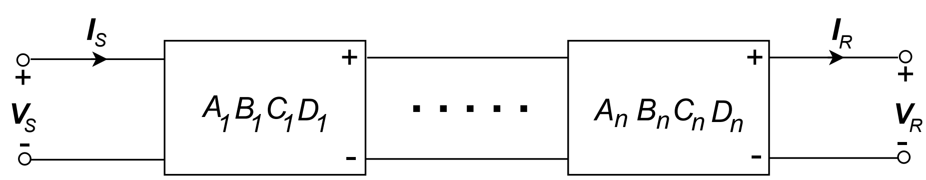


Figure 5.8: Cascaded Transmission Lines

$$\begin{bmatrix} V_S \\ I_S \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \dots \begin{bmatrix} A_n & B_n \\ C_n & D_n \end{bmatrix} \begin{bmatrix} V_R \\ I_R \end{bmatrix} \tag{5.56}$$

where the ABCD matrix for the cascaded transmission line is a multiplication of matrices for each of the individual transmission line models.

Next, we formalize the ABCD matrix for the cascaded transmission line in HOL Light as follows:

Definition 5.37. *Cascaded ABCD Matrix*

```
⊢def ∀tlms tlmpa w R L Ca G. cascaded_abcd_matrix ([ ],w) = cidentity_mat ∧
    cascaded_abcd_matrix (CONS (tlms,(R,L,Ca,G)) tlmpa,w) =
            (abcd_mat tlms (R,L,Ca,G) w) ** cascaded_abcd_matrix (tlmpa,w)
```

where the HOL Light function `cascaded_abcd_matrix` uses a variable of type `tl_models_param_all` and models the ABCD matrix of the cascaded transmission line using a recursive definition in HOL Light. We use the above definition to verify the ABCD matrix of a cascaded transmission line (medium and short transmission lines) as:

Theorem 5.23. *ABCD Matrix of Cascaded Two Port Circuit/Transmission Line*

```
⊢thm ∀R1 R2 L1 L2 Ca1 Ca2 G1 G2 w.
    let tlc1 = ((R1,L1,Ca1,G1):trans_lines_const) and
      tlc2 = ((R2,L2,Ca2,G2):trans_lines_const) in
      [A] valid_cascaded_tl ([MediumTL_PiCir,tlc1; ShortTL_SerImp,tlc2], w)
    ⇒ cascaded_abcd_matrix ([MediumTL_PiCir,tlc1; ShortTL_SerImp,tlc2], w) =
```

$$\begin{bmatrix} \texttt{T3} & \texttt{T3} * \texttt{T2} + \texttt{T1} \\ \texttt{w} * \texttt{Ca1} * \texttt{T4} & \texttt{w} * \texttt{Ca1} * \texttt{T4} * \texttt{T2} + \texttt{T3} \end{bmatrix}$$

where, $\texttt{T1} = \texttt{R1} + \texttt{ii} * \texttt{w} * \texttt{L1}$, $\texttt{T2} = \texttt{R2} + \texttt{ii} * \texttt{w} * \texttt{L2}$

$$\texttt{T3} = 1 + \frac{\texttt{w} * \texttt{Ca1} * \texttt{T1}}{2} \qquad \texttt{T4} = 1 + \frac{\texttt{w} * \texttt{Ca1} * \texttt{T1}}{4}$$

The only assumption `A` ensures the validity of the cascaded transmission line. The proof process of the above theorem is mainly based on Definition 5.37, properties of lists and matrices, alongside some complex arithmetic reasoning.

In the following subsection, we present the formal analysis of a Wireless Power Transfer system, as a real-world application, where we use our formalization of transmission lines presented earlier.

5.6.3.4 Wireless Power Transfer System

A Wireless Power Transfer (WPT) system enables the transmission of electrical energy from source to destination without establishing a physical connection [114]. A WPT system uses the phenomenon of electromagnetic fields based on the induction coils to send energy from the transmitter to the receiver. It has been widely used in Electrical Vehicles (EVs) [115] and implantable medical devices [116]. For example, it is used for charging batteries in EVs, where the placement of wires is not possible due to a restricted space. To analyze the process of the energy transmission, the WPT system is represented as a T-shape transmission line lumped model [117]. Next, the ABCD parameters are analyzed to study the relationship between the voltages and current on the sending and receiving ends. The T-shape lumped medium transmission line model for the series-series compensation WPT system is depicted in Figure 5.9 [117]. Here, R_1, C_1 and L_{k1} model the resistance, capacitance and leakage inductance on the sending end. Similarly, R_2, C_2 and L_{k2} capture the same quantities on the receiving end. M_{12} is the mutual inductance between the sending and receiving coils. Similarly, V_1, I_1, V_2 and I_2 represent the sending and receiving end voltages and currents, respectively.

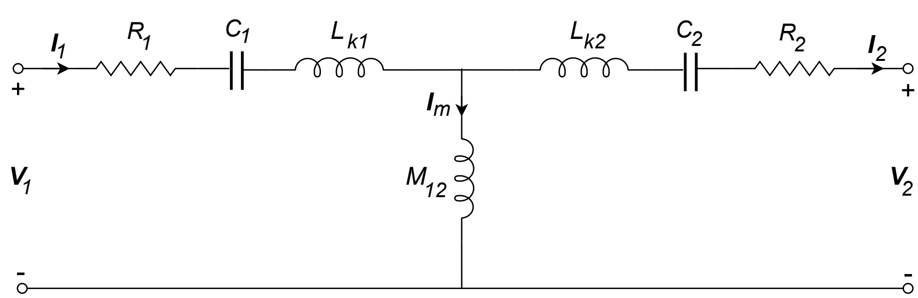


Figure 5.9: T-Model for WPT System

Here, for simplicity, we consider the sending and receiving coils are identical. Therefore, the resistance, capacitance and leakage inductance on the sending and receiving ends are equal to each other, i.e., $R_1 = R_2 = R$, $C_1 = C_2 = C$ and $L_{k1} = L_{k2} = L$. Now, we can express the relationship between the sending and receiving quantities for T-shaped lumped model for the WPT system as follows [117]:

$$\begin{bmatrix} V_1 \\ \\ I_1 \end{bmatrix} = \begin{bmatrix} 1 + \frac{K_1}{K_2} & K_1 \left(2 + \frac{K_1}{K_2}\right) \\ \frac{1}{K_2} & 1 + \frac{K_1}{K_2} \end{bmatrix} \begin{bmatrix} V_2 \\ \\ I_2 \end{bmatrix} \tag{5.57}$$

where,

$K_1 = R + jwL_k + \frac{1}{jwC}$, $K_2 = jwM_{12}$

We formally verify the above ABCD parameters based model of the WPT system (Equation (5.57)) based on the KCL and KVL implementations as the following theorem in HOL Light:

Theorem 5.24. *ABCD Matrix of WPT System*

```
⊢thm ∀Vs Is VR IR R Lk Ca G M12 w.
    let se = ((Vs,Is):send_end_quan) and
        re = ((VR,IR):recei_end_quan) and
       tlc = ((R,Lk,Ca,G):trans_lines_const) in
       [A] valid_transmission_line (MediumTL_TCir,tlc)
          ⇒ (kvl_implem_wpt MediumTL_TCir se re tlc w z ∧
                kcl_implem_wpt MediumTL_TCir se re tlc M12 w z) ⇔
                    (relat_sr_wpt MediumTL_TCir se re tlc M12 w z)
```

The proof process of the above theorem is straightforward, thanks to our formalization of ABCD parameters based models. The HOL Light script for the ABCD parameters based models of transmission lines' formalizations and proofs is available at [118].

5.7 Summary and Discussion

In this chapter, we presented a higher-order logic formalization of the telegrapher's and wave equations. We first formalized the telegrapher's equations and their alternate representations, i.e., wave equations in time and phasor domains using HOL Light. Furthermore, we verified the relationship between the telegrapher's and wave equations in the phasor domain. We then constructed the formal proof for the general solutions of the telegrapher's equations in the phasor domain. Subsequently, we proved the relation between the phasor and the time-domain functions to formally verify the general solutions for the time-domain PDEs. Finally, we presented several practical applications of the telegrapher's equations such as terminated, coupled and cascaded transmission lines. The main purpose of this work is the formal development of a transmission line theory within the sound core of a higher-order logic theorem prover to analyze transmission systems. For our constructive formalization, we first formally analyzed the variations of the line voltage and current utilizing the phasor representations of the telegrapher's equations because the phasor approach reduces the time domain PDEs to ODEs. In the verification of the ODEs, we proved lemmas about the derivatives of the general solutions. One of the main challenges of the presented work was to formally verify the general solutions for the time-domain PDEs. The process began by translating solutions from the phasor domain, where they are articulated as complex-valued functions of frequency, into the time domain as real-valued functions to establish solutions for PDEs. In the HOL Light proof process, we subsequently faced the requirement to transform the time-domain functions back into complex-valued forms. This was essential because the time-domain PDEs are defined using complex derivatives, and the challenge lays in adeptly employing these complex derivatives during the proof procedure. We also proved the necessary lemmas about the complex differentions of the general solutions with respect to the parameters z and t. In addition, we provided proofs of the attenuation and phase constants for the lossless line and some other theorems regarding exponential functions and complex numbers in order to verify the correctness of the wave solutions for the time-domain PDEs. Once we proved the required theorems and lemmas, the verification of the correctness of the equations just took several lines of proof steps. For example, the proofs of the general solutions of the wave equations for voltage and current just took 19 and 22 lines, which clearly illustrates the benefit of the formally verified lemmas and theorems. Another difficulty encountered in this formalization pertains to the considerable level of user intervention. However, we developed several tactics that automate certain parts of our proofs resulting in a reduction of the length of proof scripts in many instances (e.g., reducing part of the code by around 240 lines) and make the proofs simpler and more compact. Examples of such tactics are `SHORT_TAC` and `EQ_DIFF_SIMP`, which allowed us to simplify complex arithmetics involved in the proof of the time-domain solutions. For instance, `EQ_DIFF_SIMP` is constructed to efficiently deal with the repetitive patterns in our proof procedure by consolidating them into a single tactic. This proves to be efficient in refining and optimizing our overall approach. The main advantage of the conducted formal proofs of the telegrapher's equations is that all the underlying assumptions can be explicitly written contrary to the case of paper-and-pencil proofs and proof-steps that are mechanically verified using a theorem prover.

In terms of practical applications, the analysis of coupled transmission lines posed significant challenges, as it required an understanding of various fundamental aspects, including electromagnetic theory and microwave engineering. In particular, for those of us who are not experts in electromagnetics, it has been challenging to comprehend the formal definitions used to model transmission systems and phenomena. Another challenge encountered during this formalization was the mathematical proof itself. We relied on snippets of proofs gathered from the literature including textbooks, articles and courses. However, we frequently found these traditional pen-and-paper proofs to be somewhat incomplete or lack rigorous details. Due to the nature of the analysis, we had to develop our own proof with all necessary details for the verification process. The primary benefit of this work includes the accuracy of verified results and the revelation of hidden assumptions, which are often omitted in textbooks and engineering literature. For the ABCD parameter-based models of cascaded transmission lines, our proposed formalization included a systematic approach that is easy to follow, even for non-HOL users. One of the challenging parts was the development of the deep embedding based formalization of the transmission lines, where we needed to gather all details about the parameters that contribute to the dynamics of the two-port models of transmission lines and packaging them into our formalization. In conclusion, the formalization of the transmission line theory provides mathematicians and engineers with the ability to modify and reuse the formal library in HOL Light, in contrast to conventional manual mathematical analysis.

Chapter 6

Conclusion and Future Work

This thesis presented methodologies and libraries for the formalization of partial differential equations used to model physical systems, along with the formal verification of their analytical solutions in simple type theory. This chapter concludes the thesis with a summary of the main contributions and suggestions for future work.

6.1 Conclusion

PDEs are fundamental for modeling the mathematical laws underlying physical systems. They play a critical role in the analysis, prediction, and control of various processes by providing a formal mathematical infrastructure for relating the changes in quantities of interest with respect to multiple variables, mainly space and time. Given their role in modeling the dynamics of safety-critical systems, such as transmission lines and thermal protection, an accurate analysis of PDEs is vital. However, this analysis is often conducted using informal techniques, such as numerical simulations and manual proofs, which can lead to errors. In the past few decades, formal verification techniques have increasingly been utilized for the rigorous analysis of various engineering and physical systems. However, a comprehensive formal analysis of partial differential equations and the formal verification of their applications in safety-critical domain using proof assistants are quite rare.

In this thesis, we proposed to develop a framework utilizing higher-order logic theorem proving for the mathematical modeling, analysis, and verification of physical systems governed by partial differential equations. In particular, the framework we provided consists in a higher-order logic formalization of four partial differential equations, namely, the heat, the Laplace, the telegrapher's and the wave equations as well as the formal verification of their analytical solutions using the HOL Light theorem prover. Firstly, we focused on the formalization of the one-dimensional heat conduction problem. In particular, we formally modeled the temperature variation in an one-dimensional rectangular slab using a PDE as a heat equation and formally verified its analytical solution employing the method of separation of variables based on various boundary and initial conditions. We also formally verified the convergence of the general solution.

Secondly, we formalized the Laplace equation in both Cartesian and polar coordinates, along with standard potential flows, namely, uniform, source/sink, doublet and vortex flows. Subsequently, we formally verified the linearity of the Laplace operator which is crucial to construct more complicated flow fields by superimposing these standard potential flows. We then provided the formal verification of the validity of exact potential flow solutions for the Laplace equation for aerodynamic applications. The strength of the proposed formalization is demonstrated by conducting the formal analysis of different practical applications such as the Rankine oval, potential flow past a circular cylinder and potential flow past a rotating circular cylinder.

Finally, we developed a formal model of the telegrapher's equations and their alternative form, the wave equations, in both time and phasor domains. We also formally verified the solutions of the wave and telegrapher's equations in the phasor domain. These phasor-domain solutions were then employed to formally verify the time-domain partial differential equations for current and voltage in an electrical transmission lines. To demonstrate the practical applicability of our formalization, we conducted a formal analysis of a terminated transmission line by verifying key properties such as load impedance and voltage reflection coefficient. Additionally, we performed a formal analysis of short- and open-circuited transmission lines, which are frequently used in antenna design. Furthermore, we formally analyzed coupled and cascaded transmission lines as a more realistic applications of the telegrapher's equations.

This study was conducted using the HOL Light theorem prover, chosen primarily for its support of higher-order logic formalizations and the availability of necessary theories. Moreover, HOL Light provides some valuable automated tactics, such as `REAL_DIFF_TAC` and `COMPLEX_DIFF_TAC`, which can automatically compute the real and complex differentiation of basic functions, significantly reducing user interaction time during theorem proving. Despite some of these automated tactics, HOL Light still requires considerable manual guidance due to the undecidability nature of higher-order logic. This need for extensive user interaction represents a key limitation of the approach employed in this work. The methodologies developed in this research represent a foundational contribution to the formal development of theories relevant to practical and widely used real-world systems that are modeled by PDEs. Additionally, our approach offers potential cost reductions associated with integrating theorem proving into the critical phases of designing and verifying physical systems. Furthermore, our work can have a broader impact on future formalization efforts related to partial differential equations and their applications.

6.2 Future Work

The formalization and verification results presented in this thesis pave the way for utilizing theorem proving to conduct the precise analysis of physical systems modeled by partial differential equations, complementing traditional paper-and-pencil and simulation approaches. This work can be extended to further refine the rigorous analysis of PDEs in modeling physical problems and dynamic behaviors in critical engineering systems. In the following, we outline several potential directions for future research based on the experiences and lessons learned during this thesis:

- Formal analysis of transient heat conduction in one-dimensional composite slabs: The current model focuses on heat transfer in a rectangular slab, with the formal verification of the closed-form solution for this specific case. This work can be extended to address transient thermal problems in multi-layered structures, allowing for more comprehensive thermal analysis in the context of thermal protection systems. This extension involves applying the method of separation of variables to the heat conduction equation in each region of the solid. During this process, the thermal diffusivity of each layer is preserved in the part of the modified heat conduction equation associated with the time-dependent function, similar to the approach used for a single-region problem [119].

- Extension of formalization of standard potential flows to complex potential flows for the Laplace equation: The current formalization covers the validation of real potential flows for the Laplace equation and their applications. One can extend these real flows to complex-valued potential flows, enabling the analysis of more complicated aerodynamic problems, such as flow around aerofoils. This requires the formal proof of the Cauchy-Riemann equations [120] for both the stream function and the velocity potential, as well as the symmetry of partial derivatives (Clairaut's Theorem [15]). It is also interesting to consider the formal verification of the relationship between the Laplace equation in Cartesian and polar coordinates, which needs the development of formal theorems for the chain rule of partial derivatives.

- Extension of the formal analysis of the telegrapher's and the wave equations to account for deviations in real circuits from idealized models. The current analysis simplifies the telegrapher's and the wave equations by using idealized circuit components. A future research, one could extend this work to incorporate non-idealized models, enabling a better representation of practical applications in real-world scenarios. Real circuits often exhibit behaviors that change with frequency, such as dispersion. Addressing these would require moving to the phasor domain through Fourier analysis, which decomposes the signal into its frequency components. To obtain the complete solution, formalizing the inverse Fourier transform may also be necessary.

- Extension of the formal analysis of coupled transmission lines. The current work models coupled line behavior using the generalized telegrapher's equations and verifies these equations in the phasor domain. Future efforts could extend this verification by translating the phasor domain solutions back to the time domain and ensuring their correctness for the time-domain partial differential equations. This may require formalizing the Fourier and inverse Fourier transforms, or employing an alternative method. Additionally, the formal analysis could be expanded to investigate crosstalk in communication circuits.

- Formalization of long transmission line. The current work with ABCD parameter-based models includes the formal analysis of short, medium, and cascaded lines. In a future research, one could derive the governing telegrapher's equations for long transmission lines by applying Kirchhoff's Current Law (KCL) and Kirchhoff's Voltage Law (KVL) to the circuit and formally verify the ABCD parameters based models of long transmission line.

- Development of a formal library for existence and uniqueness theorems for the solutions of PDEs: While the current work has focused on the formal verification of closed-form solutions for practical applications, constructing such a library would provide a rigorous foundation for addressing more general theoretical aspects of PDEs. This would not only enhance the scope of formal analysis but also contribute to the verification of broader classes of PDE problems in various domains.

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- **Bio-Cf4** E. Deniz, A. Rashid, O. Hasan and S. Tahar: "On the Formalization of the Heat Conduction Problem in HOL". In: *Intelligent Computer Mathematics*, Lecture Notes in Artificial Intelligence (LNAI), Vol. 13467, pp. 21-37, Springer, 2022.