

Contents

1 CPDFT Theory

Built: 15 November 2018

Parent Theories: WCSP_shared

1.1 Definitions

[indep_CPADND_def]

```
⊢ ∀X Y Z p.  
  indep_CPADND X Y Z p ⇔  
  indep_varp p lborel (λx. real (X x))  
    (pair_measure lborel lborel)  
    (λx. (real (Y x), real (Z x))) ∧  
  indep_var p lborel (λx. real (Y x)) lborel  
    (λx. real (Z x))
```

[UNIONL_def]

```
⊢ (UNIONL [] = { }) ∧ ∀s ss. UNIONL (s::ss) = s ∪ UNIONL ss
```

1.2 Theorems

[BIGUNION_CPDFT_subevent]

```
⊢ ∀t.  
  BIGUNION  
  { {x' | 0 ≤ x' ∧ x' ≤ t ∧ real q1 < x'} ×  
    { (y', z') | 0 ≤ y' ∧ y' < real q1 ∧ z' < y' ∧ 0 ≤ z'} |  
    q1 ∈ Q_set } =  
  { (x', y', z') |  
    0 ≤ z' ∧ 0 ≤ y' ∧ 0 ≤ x' ∧ z' < y' ∧ y' < x' ∧ x' ≤ t }
```

[BIGUNION_CPDFT_subevent0]

```
⊢ ∀t q0 q1.  
  BIGUNION  
  { {x' | 0 ≤ x' ∧ x' ≤ t ∧ real q1 < x'} ×  
    { {y' | 0 ≤ y' ∧ y' < real q1 ∧ real q0 < y'} ×  
      { z' | 0 ≤ z' ∧ z' < real q0 } } |  
    q0 ∈ Q_set } =  
  { x' | 0 ≤ x' ∧ x' ≤ t ∧ real q1 < x'} ×  
  { (y', z') | 0 ≤ y' ∧ y' < real q1 ∧ z' < y' ∧ 0 ≤ z' }
```

[BIGUNION_CPDFT_subevent0_measurable_set]

```
⊢ ∀t q1.  
  { x' | 0 ≤ x' ∧ x' ≤ t ∧ real q1 < x'} ×  
  { (y', z') | 0 ≤ y' ∧ y' < real q1 ∧ z' < y' ∧ 0 ≤ z'} ∈  
  measurable_sets  
  (pair_measure lborel (pair_measure lborel lborel))
```

[CDF_def3]

$$\vdash \forall p \ X \ x. \text{prob_space } p \wedge (\forall s. 0 \leq X s) \wedge (\forall n. \text{PREIMAGE } X \{y \mid x - 1 / \&\text{SUC } n < y \wedge y \leq x\} \cap \text{p_space } p \in \text{events } p \wedge \text{PREIMAGE } X \{y \mid y \leq x\} \cap \text{p_space } p \in \text{events } p \wedge \text{PREIMAGE } X \{y \mid y < x\} \cap \text{p_space } p \in \text{events } p \wedge \text{PREIMAGE } X \{y \mid y \leq x - 1 / \&\text{SUC } n\} \cap \text{p_space } p \in \text{events } p) \wedge \text{PREIMAGE } X \{y \mid y = x\} \cap \text{p_space } p \in \text{events } p \wedge (\forall z. (\lambda x. \text{real } (\text{CDF } p \ X \ x)) \text{ contl } z) \Rightarrow (\text{CDF } p \ X \ x = \text{prob } p (\text{PREIMAGE } X \{y \mid 0 \leq y \wedge y < x\} \cap \text{p_space } p))$$

[CDF_GTO]

$$\vdash \forall p \ X \ t. (\forall s. 0 \leq X s) \Rightarrow (\text{CDF } p \ X \ t = \text{distribution } p \ X \ \{y \mid 0 \leq y \wedge y \leq t\})$$

[CDF_pos_fn_integral_indicator2]

$$\vdash \forall X \ p \ t \ M. \text{random_variable } X \ p (\text{m_space } M, \text{measurable_sets } M) \wedge (\forall s. 0 \leq X s) \wedge \text{measure_space } M \wedge (\forall t. \{y \mid 0 \leq y \wedge y \leq t\} \in \text{measurable_sets } M) \Rightarrow (\text{CDF } p \ X \ t = \text{pos_fn_integral } (\text{distr } p \ M \ X) (\lambda x. \text{indicator_fn } \{y \mid 0 \leq y \wedge y \leq t\} \ x))$$

[CDF_pos_fn_integral_indicator_gt0]

$$\vdash \forall X \ p \ t \ M. \text{random_variable } X \ p (\text{m_space } M, \text{measurable_sets } M) \wedge (\forall s. 0 \leq X s) \wedge \text{measure_space } M \wedge (\{y \mid 0 \leq y \wedge y \leq t\} \in \text{measurable_sets } M) \Rightarrow (\text{CDF } p \ X \ t = \text{pos_fn_integral } (\text{distr } p \ M \ X) (\lambda x. \text{indicator_fn } \{y \mid 0 \leq y \wedge y \leq t\} \ x))$$

[CPDFT_A_Ayz_indicator_of_indicator]

$$\vdash \forall x \ y \ z \ t. \text{indicator_fn} \{(x', y', z') \mid 0 \leq z' \wedge 0 \leq y' \wedge 0 \leq x' \wedge z' < y' \wedge y' < x' \wedge x' \leq t\} (x, y, z) = \text{indicator_fn} \{x' \mid 0 \leq x' \wedge x' \leq t\} \ x \times \text{indicator_fn} \{(y', z') \mid 0 \leq z' \wedge 0 \leq y' \wedge z' < y' \wedge y' < x\} (y, z)$$

[CPDFT_Ayz_measurable_set]

$\vdash \forall x.$
 $\{(y', z') \mid 0 \leq z' \wedge 0 \leq y' \wedge z' < y' \wedge y' < x\} \in$
 $\text{measurable_sets } (\text{pair_measure lborel lborel})$

[CPDFT_event]

$\vdash \forall X \ Y \ Z \ t \ p.$
 $\text{rv_gt0_ninfy } [X; Y; Z] \wedge$
 $(\forall s. \text{ALL_DISTINCT } [X \ s; Y \ s; Z \ s]) \Rightarrow$
 $(\text{DFT_event } p$
 $\quad (\text{D_AND } (\text{D_AND } X \ (\text{D_BEFORE } Z \ Y)) \ (\text{D_BEFORE } Y \ X)) \ t =$
 $\quad \{s \mid$
 $\quad 0 \leq Z \ s \wedge 0 \leq Y \ s \wedge 0 \leq X \ s \wedge Z \ s < Y \ s \wedge Y \ s < X \ s \wedge$
 $\quad X \ s \leq \text{Normal } t\} \cap \text{p_space } p)$

[CPDFT_event_measurable_set]

$\vdash \forall t.$
 $\{(x', y', z') \mid$
 $0 \leq z' \wedge 0 \leq y' \wedge 0 \leq x' \wedge z' < y' \wedge y' < x' \wedge x' \leq t\} \in$
 measurable_sets
 $\quad (\text{pair_measure lborel } (\text{pair_measure lborel lborel}))$

[CPDFT_reduce]

$\vdash \forall X \ Y \ Z.$
 $(\forall s. \text{ALL_DISTINCT } [X \ s; Y \ s; Z \ s]) \Rightarrow$
 $(\text{P_AND } (\text{P_AND } Z \ Y) \ X =$
 $\quad \text{D_AND } (\text{D_AND } X \ (\text{D_BEFORE } Z \ Y)) \ (\text{D_BEFORE } Y \ X))$

[CPDFT_subevent0_measurable_set]

$\vdash \forall t \ q_0 \ q_1.$
 $\{x' \mid 0 \leq x' \wedge x' \leq t \wedge \text{real } q_1 < x'\} \times$
 $(\{y' \mid 0 \leq y' \wedge y' < \text{real } q_1 \wedge \text{real } q_0 < y'\} \times$
 $\{z' \mid 0 \leq z' \wedge z' < \text{real } q_0\}) \in$
 measurable_sets
 $\quad (\text{pair_measure lborel } (\text{pair_measure lborel lborel}))$

[IN_REST]

$\vdash \forall x \ s. \ x \in \text{REST } s \iff x \in s \wedge x \neq \text{CHOICE } s$

[IN_UNIONL]

$\vdash \forall l \ v. \ v \in \text{UNIONL } l \iff \exists s. \ \text{MEM } s \ l \wedge v \in s$

[indicator_of_indicator_Ayz_Ay_Az]

$\vdash \forall y \ z \ x.$
 indicator_fn
 $\quad \{(y', z') \mid 0 \leq z' \wedge 0 \leq y' \wedge z' < y' \wedge y' < x\} \ (y, z) =$
 $\text{indicator_fn } \{y' \mid 0 \leq y' \wedge y' < x\} \ y \times$
 $\text{indicator_fn } \{z' \mid 0 \leq z' \wedge z' < y\} \ z$

[indicator_of_indicator_CPDFT]

```

 $\vdash \forall x \ y \ z \ t.$ 
  indicator_fn
  { (x',y',z') |
    0 \leq z' \wedge 0 \leq y' \wedge 0 \leq x' \wedge z' < y' \wedge y' < x' \wedge x' \leq t }
  (x,y,z) =
  indicator_fn { x' | 0 \leq x' \wedge x' \leq t } x \times
  indicator_fn { y' | 0 \leq y' \wedge y' < x } y \times
  indicator_fn { z' | 0 \leq z' \wedge z' < y } z

```

[measure_space_distr_pair]

```

 $\vdash \forall X \ Y \ p \ M_1 \ M_2.$ 
  measure_space M_1 \wedge measure_space M_2 \wedge
  sigma_finite_measure M_2 \wedge
  random_variable X p (m_space M_1,measurable_sets M_1) \wedge
  random_variable Y p (m_space M_2,measurable_sets M_2) \Rightarrow
  measure_space
  (distr p (pair_measure M_1 M_2) (\lambda x. (X x,Y x)))

```

[PREIMAGE_3rv_EXTREAL_REAL_CPDFT]

```

 $\vdash \forall X \ Y \ Z \ p \ t.$ 
  rv_gt0_ninfty [X; Y; Z] \wedge 0 \leq t \Rightarrow
  (PREIMAGE (\lambda s. (X s,Y s,Z s))
  { (x,y,z) |
    0 \leq z \wedge 0 \leq y \wedge 0 \leq x \wedge z < y \wedge y < x \wedge x \leq \text{Normal } t } \cap
  p_space p =
  PREIMAGE (\lambda s. (real (X s),real (Y s),real (Z s)))
  { (x',y',z') |
    0 \leq z' \wedge 0 \leq y' \wedge 0 \leq x' \wedge z' < y' \wedge y' < x' \wedge x' \leq t } \cap
  p_space p)

```

[PREIMAGE_CPDFT_event]

```

 $\vdash \forall X \ Y \ Z \ t \ p.$ 
  rv_gt0_ninfty [X; Y; Z] \wedge
  (\forall s. ALL_DISTINCT [X s; Y s; Z s]) \Rightarrow
  (DFT_event p
  (D_AND (D_AND X (D_BEFORE Z Y)) (D_BEFORE Y X)) t =
  PREIMAGE (\lambda s. (X s,Y s,Z s))
  { (x,y,z) |
    0 \leq z \wedge 0 \leq y \wedge 0 \leq x \wedge z < y \wedge y < x \wedge x \leq \text{Normal } t } \cap
  p_space p)

```

[prob_CPDFT]

```

 $\vdash \forall p \ X \ Y \ Z \ t \ f_y \ f_x.$ 
  prob_space p \wedge 0 \leq t \wedge indep_CPAND X Y Z p \wedge
  rv_gt0_ninfty [X; Y; Z] \wedge
  (\forall s. ALL_DISTINCT [X s; Y s; Z s]) \wedge
  distributed p lborel (\lambda x. real (X x)) f_x \wedge

```

```


$$\begin{aligned}
& (\forall x. 0 \leq f_x x) \wedge \\
& \text{distributed } p \text{ lborel } (\lambda x. \text{real } (Y x)) f_y \wedge \\
& (\forall y. 0 \leq f_y y) \wedge \text{cont\_CDF } p \text{ } (\lambda x. \text{real } (Z x)) \Rightarrow \\
& (\text{prob } p \text{ (DFT\_event } p \text{ (P\_AND (P\_AND } Z Y) X) t) = \\
& \quad \text{pos\_fn\_integral lborel} \\
& \quad (\lambda x. \\
& \quad \quad f_x x \times \text{indicator\_fn } \{x' \mid 0 \leq x' \wedge x' \leq t\} x \times \\
& \quad \quad \text{pos\_fn\_integral lborel} \\
& \quad \quad (\lambda y. \\
& \quad \quad \quad f_y y \times \\
& \quad \quad \quad \text{indicator\_fn } \{y' \mid 0 \leq y' \wedge y' < x\} y \times \\
& \quad \quad \quad \text{CDF } p \text{ } (\lambda x. \text{real } (Z x)) y)))
\end{aligned}$$


```

[prob_CPDFT_initial]

```


$$\begin{aligned}
& \vdash \forall X Y Z p t. \\
& \quad \text{prob\_space } p \wedge 0 \leq t \wedge \\
& \quad \text{indep\_varp } p \text{ lborel } (\lambda x. \text{real } (X x)) \\
& \quad (\text{pair\_measure lborel lborel}) \\
& \quad (\lambda x. (\text{real } (Y x), \text{real } (Z x))) \wedge \\
& \quad \text{rv\_gt0\_ninfy } [X; Y; Z] \wedge \\
& \quad (\forall s. \text{ALL\_DISTINCT } [X s; Y s; Z s]) \wedge \\
& \quad \{(x', y', z') \mid \\
& \quad 0 \leq z' \wedge 0 \leq y' \wedge 0 \leq x' \wedge z' < y' \wedge y' < x' \wedge x' \leq t\} \in \\
& \quad \text{measurable\_sets} \\
& \quad (\text{pair\_measure lborel } (\text{pair\_measure lborel lborel})) \Rightarrow \\
& \quad (\text{prob } p \\
& \quad (\text{DFT\_event } p \\
& \quad (\text{D\_AND (D\_AND } X \text{ (D\_BEFORE } Z Y)) \text{ (D\_BEFORE } Y X)) t) = \\
& \quad \text{pos\_fn\_integral } (\text{distr } p \text{ lborel } (\lambda x. \text{real } (X x))) \\
& \quad (\lambda x. \\
& \quad \quad \text{pos\_fn\_integral} \\
& \quad \quad (\text{distr } p \text{ } (\text{pair\_measure lborel lborel}) \\
& \quad \quad (\lambda x. (\text{real } (Y x), \text{real } (Z x)))) \\
& \quad \quad (\lambda (y, z). \\
& \quad \quad \quad \text{indicator\_fn} \\
& \quad \quad \quad \{(x', y', z') \mid \\
& \quad \quad \quad 0 \leq z' \wedge 0 \leq y' \wedge 0 \leq x' \wedge z' < y' \wedge \\
& \quad \quad \quad y' < x' \wedge x' \leq t\} (x, y, z))))
\end{aligned}$$


```

[prob_CPDFT_initial2]

```


$$\begin{aligned}
& \vdash \forall X Y Z A p f_x. \\
& \quad \text{prob\_space } p \wedge \\
& \quad \text{random\_variable } (\lambda x. \text{real } (X x)) p \\
& \quad (\text{m\_space lborel}, \text{measurable\_sets lborel}) \wedge \\
& \quad \text{random\_variable } (\lambda x. \text{real } (Y x)) p \\
& \quad (\text{m\_space lborel}, \text{measurable\_sets lborel}) \wedge \\
& \quad \text{random\_variable } (\lambda x. \text{real } (Z x)) p \\
& \quad (\text{m\_space lborel}, \text{measurable\_sets lborel}) \wedge \\
& \quad A \in
\end{aligned}$$


```

```

measurable_sets
  (pair_measure lborel (pair_measure lborel lborel)) ∧
distributed p lborel (λx. real (X x)) f_x ∧
(∀x. 0 ≤ f_x x) ⇒
(pos_fn_integral (distr p lborel (λx. real (X x)))
  (λx.
    pos_fn_integral
      (distr p (pair_measure lborel lborel)
        (λx. (real (Y x), real (Z x))))
        (λ(y,z). indicator_fn A (x,y,z))) =
pos_fn_integral lborel
  (λx.
    f_x x ×
    pos_fn_integral
      (distr p (pair_measure lborel lborel)
        (λx. (real (Y x), real (Z x))))
        (λ(y,z). indicator_fn A (x,y,z))))
```

[prob_CPDFT_initial3]

```

⊢ ∀Y Z f_x p t.
prob_space p ∧
random_variable (λx. real (Y x)) p
  (m_space lborel, measurable_sets lborel) ∧
random_variable (λx. real (Z x)) p
  (m_space lborel, measurable_sets lborel) ⇒
(pos_fn_integral lborel
  (λx.
    f_x x ×
    pos_fn_integral
      (distr p (pair_measure lborel lborel)
        (λx. (real (Y x), real (Z x))))
        (λ(y,z).
          indicator_fn
            {(x',y',z') |
              0 ≤ z' ∧ 0 ≤ y' ∧ 0 ≤ x' ∧ z' < y' ∧
              y' < x' ∧ x' ≤ t} (x,y,z))) =
pos_fn_integral lborel
  (λx.
    f_x x × indicator_fn {x' | 0 ≤ x' ∧ x' ≤ t} x ×
    pos_fn_integral
      (distr p (pair_measure lborel lborel)
        (λx. (real (Y x), real (Z x))))
        (λ(y,z).
          indicator_fn
            {(y',z') |
              0 ≤ z' ∧ 0 ≤ y' ∧ z' < y' ∧ y' < x}
              (y,z))))
```

[prob_CPDFT_initial4]

```

 $\vdash \forall p \ Y \ Z \ x \ f\_y.$ 
 $\text{indep\_var } p \text{ lborel } (\lambda x. \text{ real } (Y \ x)) \text{ lborel}$ 
 $\quad (\lambda x. \text{ real } (Z \ x)) \wedge$ 
 $\text{distributed } p \text{ lborel } (\lambda x. \text{ real } (Y \ x)) \ f\_y \wedge$ 
 $\text{rv\_gt0\_ninfy } [Z] \wedge (\forall x. 0 \leq f\_y \ x) \wedge$ 
 $(\forall z. (\lambda x. \text{ real } (\text{CDF } p \ (\lambda x. \text{ real } (Z \ x)) \ x)) \text{ contl } z) \Rightarrow$ 
 $(\text{pos\_fn\_integral}$ 
 $\quad (\text{distr } p \ (\text{pair\_measure lborel lborel})$ 
 $\quad \quad (\lambda x. (\text{real } (Y \ x), \text{real } (Z \ x))))$ 
 $\quad (\lambda (y,z).$ 
 $\quad \quad \text{indicator\_fn}$ 
 $\quad \quad \{(y',z') \mid 0 \leq z' \wedge 0 \leq y' \wedge z' < y' \wedge y' < x\}$ 
 $\quad \quad (y,z)\} =$ 
 $\quad \quad \text{pos\_fn\_integral lborel}$ 
 $\quad \quad (\lambda y.$ 
 $\quad \quad \quad f\_y \ y \times \text{indicator\_fn } \{y' \mid 0 \leq y' \wedge y' < x\} \ y \times$ 
 $\quad \quad \quad \text{CDF } p \ (\lambda x. \text{ real } (Z \ x)) \ y))$ 

```

[prob_event_3rv_2indep_thd]

```

 $\vdash \forall p \ X \ Y \ Z \ M_1 \ M_2 \ A.$ 
 $\text{prob\_space } p \wedge \text{measure\_space } M_1 \wedge \text{measure\_space } M_2 \wedge$ 
 $\text{sigma\_finite\_measure } M_1 \wedge \text{sigma\_finite\_measure } M_2 \wedge$ 
 $\text{indep\_varp } p \ M_1 \ X \ M_2 \ (\lambda x. (Y \ x, Z \ x)) \wedge$ 
 $A \in \text{measurable\_sets } (\text{pair\_measure } M_1 \ M_2) \Rightarrow$ 
 $(\text{prob } p \ (\text{PREIMAGE } (\lambda x. (X \ x, Y \ x, Z \ x)) \ A) \cap \text{p\_space } p) =$ 
 $\text{pos\_fn\_integral } (\text{distr } p \ M_1 \ X)$ 
 $\quad (\lambda x.$ 
 $\quad \quad \text{pos\_fn\_integral } (\text{distr } p \ M_2 \ (\lambda x. (Y \ x, Z \ x)))$ 
 $\quad \quad (\lambda (y,z). \text{indicator\_fn } A \ (x,y,z)))$ 

```

[prob_event_le_gt0]

```

 $\vdash \forall X \ p \ t \ M.$ 
 $\text{random\_variable } X \ p \ (\text{m\_space } M, \text{measurable\_sets } M) \wedge$ 
 $\text{measure\_space } M \wedge \{y \mid 0 \leq y \wedge y \leq t\} \in \text{measurable\_sets } M \Rightarrow$ 
 $(\text{prob } p \ (\text{PREIMAGE } X \ \{y \mid 0 \leq y \wedge y \leq t\}) \cap \text{p\_space } p) =$ 
 $\text{integral } (\text{distr } p \ M \ X)$ 
 $\quad (\text{indicator\_fn } \{y \mid 0 \leq y \wedge y \leq t\}))$ 

```

[real_indicator_fn_pos_le]

```

 $\vdash \forall A \ s. 0 \leq \text{real } (\text{indicator\_fn } A \ s)$ 

```

