

Contents

1 WCSP Theory

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Parent Theories: DFT_Gates_def_PROB

1.1 Definitions

[cond_density_def]

$$\begin{aligned} \vdash \forall M_1 M_2 p X Y y f_{xy} f_y f_{cond}. \\ \text{cond_density } M_1 M_2 p X Y y f_{xy} f_y f_{cond} \iff \\ \text{random_variable } X p (\text{m_space } M_1, \text{measurable_sets } M_1) \wedge \\ \text{random_variable } Y p (\text{m_space } M_2, \text{measurable_sets } M_2) \wedge \\ \text{distributed } p (\text{pair_measure } M_1 M_2) (\lambda x. (X x, Y x)) f_{xy} \wedge \\ \text{distributed } p M_2 Y f_y \wedge \\ \forall s. f_y s \neq 0 \Rightarrow (f_{cond} y = (\lambda x. f_{xy} (x, y) / f_y y)) \end{aligned}$$

[den_gt0_ninfinity_def]

$$\begin{aligned} \vdash \forall f_{xy} f_y f_{cond}. \\ \text{den_gt0_ninfinity } f_{xy} f_y f_{cond} \iff \\ \forall x y. \\ 0 \leq f_{xy} (x, y) \wedge 0 < f_y y \wedge f_y y \neq \text{PosInf} \wedge \\ 0 \leq f_{cond} y x \end{aligned}$$

[DISJOINT_WSP_def]

$$\begin{aligned} \vdash \forall Y X_a X_d t. \\ \text{DISJOINT_WSP } Y X_a X_d t \iff \\ \text{DISJOINT} \\ \{s \mid \\ Y s < X_a s \wedge X_a s \leq \text{Normal } t \wedge 0 \leq Y s \wedge \\ Y s \leq \text{Normal } t\} \{s \mid X_d s < Y s \wedge Y s \leq \text{Normal } t\} \end{aligned}$$

[UNIONL_def]

$$\vdash (\text{UNIONL } [] = \{\}) \wedge \forall s ss. \text{UNIONL } (s :: ss) = s \cup \text{UNIONL } ss$$

1.2 Theorems

[cond_density_joint_marginal]

$$\begin{aligned} \vdash \forall M_1 M_2 p X Y f_{cond} x y f_{xy} f_x f_y. \\ (\forall y. f_y y \neq 0 \wedge f_y y \neq \text{PosInf} \wedge f_y y \neq \text{NegInf}) \wedge \\ \text{cond_density } M_1 M_2 p X Y y f_{xy} f_y f_{cond} \Rightarrow \\ (f_{xy} (x, y) = f_{cond} y x \times f_y y) \end{aligned}$$

[CSP_event]

$$\begin{aligned} \vdash \forall X Y p t. \\ (\forall s. 0 \leq Y s) \Rightarrow \\ (\text{DFT_event } p (\text{CSP } Y X) t = \\ \{s \mid \\ Y s < X s \wedge X s \leq \text{Normal } t \wedge 0 \leq Y s \wedge Y s \leq \text{Normal } t\} \cap \\ \text{p_space } p) \end{aligned}$$

[CSP_indicator_of_indicator]

$$\vdash \forall x \ y \ t. \text{indicator_fn} \{(w, u) \mid u < w \wedge w \leq t \wedge 0 \leq u \wedge u \leq t\} \\ (x, y) = \text{indicator_fn} \{w \mid y < w \wedge w \leq t\} x \times \\ \text{indicator_fn} \{u \mid 0 \leq u \wedge u \leq t\} y$$

[CSP_PREIMAGE_event_GTO]

$$\vdash \forall X \ Y \ p \ t. \text{rv_gt0_ninfinity} [X; Y] \wedge 0 \leq t \Rightarrow \\ (\text{DFT_event } p \ (\text{CSP } Y \ X) \ t = \text{PREIMAGE} (\lambda s. (\text{real} (X \ s), \text{real} (Y \ s))) \\ \{(x, y) \mid y < x \wedge x \leq t \wedge 0 \leq y \wedge y \leq t\} \cap \text{p_space } p)$$

[CSP_set_BIGUNION_IN_MEASURABLE_SETS]

$$\vdash \forall t \ q. \{w \mid \text{real } q < w \wedge w \leq t\} \times \\ \{u \mid u < \text{real } q \wedge 0 \leq u \wedge u \leq t\} \in \\ \text{measurable_sets} \ (\text{pair_measure lborel lborel})$$

[CSP_set_BIGUNION_Q]

$$\vdash \forall t. \text{BIGUNION} \\ \{\{w \mid \text{real } q < w \wedge w \leq t\} \times \\ \{u \mid u < \text{real } q \wedge 0 \leq u \wedge u \leq t\} \mid \\ q \in \text{Q_set}\} = \\ \{(w, u) \mid u < w \wedge w \leq t \wedge 0 \leq u \wedge u \leq t\}$$

[CSP_set_IN_MEASURABLE_SETS_PAIR_lborel]

$$\vdash \forall t. \{(w, u) \mid u < w \wedge w \leq t \wedge 0 \leq u \wedge u \leq t\} \in \\ \text{measurable_sets} \ (\text{pair_measure lborel lborel})$$

[DFT_CSP_event_GTO]

$$\vdash \forall X \ Y \ p \ t. (\forall s. 0 \leq Y \ s) \Rightarrow \\ (\text{DFT_event } p \ (\text{CSP } Y \ X) \ t = \text{PREIMAGE} (\lambda s. (X \ s, Y \ s))) \\ \{(x, y) \mid y < x \wedge x \leq \text{Normal } t \wedge 0 \leq y \wedge y \leq \text{Normal } t\} \cap \\ \text{p_space } p)$$

[DFT_event_NEVER]

$$\vdash \forall p \ t. \text{DFT_event } p \ \text{NEVER } t = \{\}$$

[DISJOINT_WSP_DISJOINT_P]

$$\vdash \forall Y \ X_a \ X_d \ t \ p. \text{DISJOINT_WSP } Y \ X_a \ X_d \ t \Rightarrow \text{DISJOINT} \\ (\{s \mid Y \ s < X_a \ s \wedge X_a \ s \leq \text{Normal } t \wedge 0 \leq Y \ s \wedge Y \ s \leq \text{Normal } t\} \cap \text{p_space } p) \\ (\{s \mid X_d \ s < Y \ s \wedge Y \ s \leq \text{Normal } t\} \cap \text{p_space } p)$$

[extreal_mul_linv]

$$\vdash \forall x. \ x \neq \text{PosInf} \wedge x \neq \text{NegInf} \wedge x \neq 0 \Rightarrow (\text{inv } x \times x = 1)$$

[extreal_mul_rinv]

$$\vdash \forall x. \ x \neq \text{PosInf} \wedge x \neq \text{NegInf} \wedge x \neq 0 \Rightarrow (x \times \text{inv } x = 1)$$

[IN_REST]

$$\vdash \forall x \ s. \ x \in \text{REST } s \iff x \in s \wedge x \neq \text{CHOICE } s$$

[IN_UNIONL]

$$\vdash \forall l \ v. \ v \in \text{UNIONL } l \iff \exists s. \ \text{MEM } s \ l \wedge v \in s$$

[lemma1_fun_mul_fun]

$$\vdash \forall M \ f \ f_y. \text{measure_space } M \wedge (\forall x. \ x \in \text{m_space } M \Rightarrow 0 \leq f \ x) \wedge \\ (\forall y. \ f_y \ y \neq \text{PosInf} \wedge 0 \leq f_y \ y) \Rightarrow \\ \forall y. \text{pos_fn_integral } M (\lambda x. \text{Normal} (\text{real} (f_y \ y)) \times f \ x) = \\ \text{Normal} (\text{real} (f_y \ y)) \times \text{pos_fn_integral } M (\lambda x. f \ x)$$

[lemma1_fun_mul_fun1]

$$\vdash \forall M \ f_1 \ f_2 \ f_y. \text{measure_space } M \wedge (\forall x. \ x \in \text{m_space } M \Rightarrow 0 \leq f_1 \ x) \wedge \\ (\forall x. \ x \in \text{m_space } M \Rightarrow 0 \leq f_2 \ x) \wedge \\ (\forall y. \ f_y \ y \neq \text{PosInf} \wedge 0 \leq f_y \ y) \Rightarrow \\ \forall y. \text{pos_fn_integral } M \\ (\lambda x. \text{Normal} (\text{real} (f_y \ y)) \times f_1 \ x \times f_2 \ x) = \\ \text{Normal} (\text{real} (f_y \ y)) \times \\ \text{pos_fn_integral } M (\lambda x. f_1 \ x \times f_2 \ x)$$

[lemma3_fun_mul_fun]

$$\vdash \forall M \ f \ f_y. \text{measure_space } M \wedge (\forall x. \ x \in \text{m_space } M \Rightarrow 0 \leq f \ x) \wedge \\ (\forall y. \ f_y \ y \neq \text{PosInf} \wedge 0 \leq f_y \ y) \Rightarrow \\ \forall y. \text{pos_fn_integral } M (\lambda x. f_y \ y \times f \ x) = \\ f_y \ y \times \text{pos_fn_integral } M (\lambda x. f \ x)$$

[lemma4_fun_mul_fun]

$$\vdash \forall M f f_{-y}.$$

$$\text{measure_space } M \wedge (\forall x y. x \in \text{m_space } M \Rightarrow 0 \leq f y x) \wedge$$

$$(\forall y. f_{-y} y \neq \text{PosInf} \wedge 0 \leq f_{-y} y) \Rightarrow$$

$$\forall y.$$

$$\text{pos_fn_integral } M (\lambda x. f_{-y} y \times f y x) =$$

$$f_{-y} y \times \text{pos_fn_integral } M (\lambda x. f y x)$$

[lemma5_fun_mul_fun]

$$\vdash \forall M f f_{-y} f_{-x}.$$

$$(\forall x. x \in \text{m_space } M) \wedge \text{measure_space } M \wedge$$

$$(\forall x y. x \in \text{m_space } M \Rightarrow 0 \leq f y x) \wedge$$

$$(\forall x. x \in \text{m_space } M \Rightarrow 0 \leq f_{-x} x) \wedge$$

$$(\forall y. f_{-y} y \neq \text{PosInf} \wedge 0 \leq f_{-y} y) \Rightarrow$$

$$\forall y.$$

$$\text{pos_fn_integral } M (\lambda x. f_{-y} y \times f_{-x} x \times f y x) =$$

$$f_{-y} y \times \text{pos_fn_integral } M (\lambda x. f_{-x} x \times f y x)$$

[pos_fn_integral_fun_muli]

$$\vdash \forall M Y f.$$

$$\text{measure_space } M \wedge (\forall x. x \in \text{m_space } M \Rightarrow 0 \leq f x) \wedge$$

$$(\forall y. 0 \leq Y y) \Rightarrow$$

$$\forall y.$$

$$\text{pos_fn_integral } M (\lambda x. \text{Normal } (Y y) \times f x) =$$

$$\text{Normal } (Y y) \times \text{pos_fn_integral } M (\lambda x. f x)$$

[PREIMAGE_EXTREAL_REAL_2RV_BEFORE_CSP]

$$\vdash \forall X Y t p.$$

$$(\forall s. X s \neq \text{PosInf} \wedge 0 \leq X s \wedge Y s \neq \text{PosInf} \wedge 0 \leq Y s) \wedge$$

$$0 \leq t \Rightarrow$$

$$(\text{PREIMAGE } (\lambda s. (X s, Y s)))$$

$$\{(x, y) \mid y < x \wedge x \leq \text{Normal } t \wedge 0 \leq y \wedge y \leq \text{Normal } t\} \cap$$

$$\text{p_space } p =$$

$$\text{PREIMAGE } (\lambda s. (\text{real } (X s), \text{real } (Y s)))$$

$$\{(x, y) \mid y < x \wedge x \leq t \wedge 0 \leq y \wedge y \leq t\} \cap \text{p_space } p)$$

[PREIMAGE_EXTREAL_REAL_2RV_WSP]

$$\vdash \forall X_a Y t p.$$

$$(\forall s. X_a s \neq \text{PosInf} \wedge 0 \leq X_a s \wedge Y s \neq \text{PosInf} \wedge 0 \leq Y s) \wedge$$

$$0 \leq t \Rightarrow$$

$$(\text{PREIMAGE } (\lambda s. (X_a s, Y s)))$$

$$\{(w, u) \mid u < w \wedge w \leq \text{Normal } t \wedge 0 \leq u \wedge u \leq \text{Normal } t\} \cap$$

$$\text{p_space } p =$$

$$\text{PREIMAGE } (\lambda s. (\text{real } (X_a s), \text{real } (Y s)))$$

$$\{(w, u) \mid u < w \wedge w \leq t \wedge 0 \leq u \wedge u \leq t\} \cap \text{p_space } p)$$

[prob_CSP]

```

 $\vdash \forall p X Y f_{xy} f_y f_{cond} t.$ 
 $\text{rv\_gt0\_ninf} [X; Y] \wedge 0 \leq t \wedge$ 
 $(\forall x y.$ 
 $\quad \text{cond\_density lborel lborel } p (\lambda s. \text{real } (X s))$ 
 $\quad (\lambda s. \text{real } (Y s)) y f_{xy} f_y f_{cond}) \wedge$ 
 $\text{prob\_space } p \wedge \text{den\_gt0\_ninf} f_{xy} f_y f_{cond} \Rightarrow$ 
 $(\text{prob } p (\text{DFT\_event } p (\text{CSP } Y X) t) =$ 
 $\quad \text{pos\_fn\_integral lborel}$ 
 $\quad (\lambda y.$ 
 $\quad \quad \text{indicator\_fn } \{u \mid 0 \leq u \wedge u \leq t\} y \times f_y y \times$ 
 $\quad \quad \text{pos\_fn\_integral lborel}$ 
 $\quad \quad (\lambda x.$ 
 $\quad \quad \quad \text{indicator\_fn } \{w \mid y < w \wedge w \leq t\} x \times$ 
 $\quad \quad \quad f_{cond} y x)))$ 

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[prob_CSP_initial]

```

 $\vdash \forall p X Y f_{xy} f_x f_y f_{cond} A t.$ 
 $\text{rv\_gt0\_ninf} [X; Y] \wedge 0 \leq t \wedge$ 
 $(\forall x y.$ 
 $\quad \text{cond\_density lborel lborel } p (\lambda s. \text{real } (X s))$ 
 $\quad (\lambda s. \text{real } (Y s)) y f_{xy} f_y f_{cond}) \wedge$ 
 $\text{prob\_space } p \wedge (\forall x. 0 \leq f_{xy} x) \wedge$ 
 $(\forall y. 0 < f_y y \wedge f_y y \neq \text{PosInf}) \wedge$ 
 $(\forall t.$ 
 $\quad \{ (w, u) \mid u < w \wedge w \leq t \wedge 0 \leq u \wedge u \leq t \} \in$ 
 $\quad \text{measurable\_sets } (\text{pair\_measure lborel lborel})) \Rightarrow$ 
 $(\text{prob } p (\text{DFT\_event } p (\text{CSP } Y X) t) =$ 
 $\quad \text{pos\_fn\_integral lborel}$ 
 $\quad (\lambda y.$ 
 $\quad \quad \text{pos\_fn\_integral lborel}$ 
 $\quad \quad (\lambda x.$ 
 $\quad \quad \quad \text{indicator\_fn}$ 
 $\quad \quad \quad \{ (w, u) \mid u < w \wedge w \leq t \wedge 0 \leq u \wedge u \leq t \}$ 
 $\quad \quad \quad (x, y) \times f_{cond} y x \times f_y y)))$ 

```

[prob_CSP_initial_2]

```

 $\vdash \forall p X Y f_{xy} f_y f_{cond} t.$ 
 $\text{rv\_gt0\_ninf} [X; Y] \wedge 0 \leq t \wedge$ 
 $(\forall x y.$ 
 $\quad \text{cond\_density lborel lborel } p (\lambda s. \text{real } (X s))$ 
 $\quad (\lambda s. \text{real } (Y s)) y f_{xy} f_y f_{cond}) \wedge$ 
 $\text{prob\_space } p \wedge (\forall x. 0 \leq f_{xy} x) \wedge$ 
 $(\forall y. 0 < f_y y \wedge f_y y \neq \text{PosInf}) \wedge$ 
 $(\forall t.$ 
 $\quad \{ (w, u) \mid u < w \wedge w \leq t \wedge 0 \leq u \wedge u \leq t \} \in$ 
 $\quad \text{measurable\_sets } (\text{pair\_measure lborel lborel}) \wedge$ 
 $\quad (\forall x y. 0 \leq f_{cond} y x) \Rightarrow$ 

```

```
(prob p (DFT_event p (CSP Y X) t) =
  pos_fn_integral lborel
  ( $\lambda y.$ 
    indicator_fn { $u \mid 0 \leq u \wedge u \leq t\}$  y  $\times f_y y \times$ 
    pos_fn_integral lborel
    ( $\lambda x.$ 
      indicator_fn { $w \mid y < w \wedge w \leq t\}$  x  $\times$ 
      f_cond y x)))
```

[prob_joint_conditional_density]

```
 $\vdash \forall p X Y f_{xy} f_y f_{cond} A.$ 
  ( $\forall y.$  cond_density lborel lborel p X Y y f_xy f_y f_cond)  $\wedge$ 
  prob_space p  $\wedge$  ( $\forall x.$   $0 \leq f_{xy} x$ )  $\wedge$ 
  ( $\forall y.$   $0 < f_y y \wedge f_y y \neq \text{PosInf}$ )  $\wedge$ 
  A ∈ measurable_sets (pair_measure lborel lborel)  $\Rightarrow$ 
  (prob p (PREIMAGE ( $\lambda x.$  (X x, Y x)) A ∩ p_space p) =
  pos_fn_integral lborel
  ( $\lambda y.$ 
    pos_fn_integral lborel
    ( $\lambda x.$  indicator_fn A (x, y)  $\times f_{cond} y x \times f_y y))))$ 
```

[prob_joint_density]

```
 $\vdash \forall p X Y f_{xy} A.$ 
  distributed p (pair_measure lborel lborel)
  ( $\lambda x.$  (X x, Y x)) f_xy  $\wedge$  prob_space p  $\wedge$ 
  ( $\forall x.$   $0 \leq f_{xy} x$ )  $\wedge$ 
  A ∈ measurable_sets (pair_measure lborel lborel)  $\Rightarrow$ 
  (prob p (PREIMAGE ( $\lambda x.$  (X x, Y x)) A ∩ p_space p) =
  pos_fn_integral (pair_measure lborel lborel)
  ( $\lambda x.$  indicator_fn A x  $\times f_{xy} x)))$ 
```

[prob_joint_density_iterated_integrals]

```
 $\vdash \forall p X Y f_{xy} A.$ 
  distributed p (pair_measure lborel lborel)
  ( $\lambda x.$  (X x, Y x)) f_xy  $\wedge$  prob_space p  $\wedge$ 
  ( $\forall x.$   $0 \leq f_{xy} x$ )  $\wedge$ 
  A ∈ measurable_sets (pair_measure lborel lborel)  $\Rightarrow$ 
  (prob p (PREIMAGE ( $\lambda x.$  (X x, Y x)) A ∩ p_space p) =
  pos_fn_integral lborel
  ( $\lambda y.$ 
    pos_fn_integral lborel
    ( $\lambda x.$  indicator_fn A (x, y)  $\times f_{xy} (x, y))))))$ 
```

[prob_WSP]

```
 $\vdash \forall p Y X_a X_d t f_y f_{xy} f_{cond}.$ 
  prob_space p  $\wedge$  ( $\forall s.$  ALL_DISTINCT [X_a s; X_d s; Y s])  $\wedge$ 
  DISJOINT_WSP Y X_a X_d t  $\wedge$ 
  rv_gt0_ninfinity [X_a; X_d; Y]  $\wedge$   $0 \leq t \wedge$ 
```

```
( $\forall x \ y.$ 
  cond_density lborel lborel p ( $\lambda s.$  real ( $X_a$  s)))
  ( $\lambda s.$  real ( $Y$  s))  $y f_{xy} f_y f_{cond}$ )  $\wedge$ 
  den_gt0_ninfinity  $f_{xy} f_y f_{cond}$   $\wedge$ 
  indep_var p lborel ( $\lambda s.$  real ( $X_d$  s)) lborel
  ( $\lambda s.$  real ( $Y$  s))  $\wedge$  cont_CDF p ( $\lambda s.$  real ( $X_d$  s))  $\wedge$ 
  measurable_CDF p ( $\lambda s.$  real ( $X_d$  s))  $\Rightarrow$ 
  (prob p (DFT_event p (WSP  $Y X_a X_d$ ) t) =
    pos_fn_integral lborel
    ( $\lambda y.$ 
      indicator_fn  $\{u \mid 0 \leq u \wedge u \leq t\}$   $y \times f_y y \times$ 
      pos_fn_integral lborel
      ( $\lambda x.$ 
        indicator_fn  $\{w \mid y < w \wedge w \leq t\}$   $x \times$ 
         $f_{cond} y x$ ) +
      pos_fn_integral lborel
      ( $\lambda y.$ 
         $f_y y \times$ 
        (indicator_fn  $\{u \mid 0 \leq u \wedge u \leq t\}$   $y \times$ 
        CDF p ( $\lambda s.$  real ( $X_d$  s))  $y$ )))
```

[prob_WSP_initial1]

```
 $\vdash \forall p \ Y X_a X_d t.$ 
  prob_space p  $\wedge$  ( $\forall s.$   $0 \leq Y$  s)  $\wedge$ 
  ( $\forall s.$  ALL_DISTINCT [ $X_a$  s;  $X_d$  s;  $Y$  s])  $\wedge$ 
  DISJOINT
  ( $\{s \mid$ 
     $Y_s < X_a s \wedge X_a s \leq \text{Normal } t \wedge 0 \leq Y_s \wedge$ 
     $Y_s \leq \text{Normal } t\} \cap p_{\text{space}} p$ )
  ( $\{s \mid X_d s < Y_s \wedge Y_s \leq \text{Normal } t\} \cap p_{\text{space}} p$ )  $\wedge$ 
  ( $\{s \mid$ 
     $Y_s < X_a s \wedge X_a s \leq \text{Normal } t \wedge 0 \leq Y_s \wedge Y_s \leq \text{Normal } t\} \cap$ 
     $p_{\text{space}} p \in \text{events } p \wedge$ 
    ( $\{s \mid X_d s < Y_s \wedge Y_s \leq \text{Normal } t\} \cap p_{\text{space}} p \in \text{events } p$ )  $\Rightarrow$ 
    (prob p (DFT_event p (WSP  $Y X_a X_d$ ) t) =
      prob p
      ( $\{s \mid$ 
         $Y_s < X_a s \wedge X_a s \leq \text{Normal } t \wedge 0 \leq Y_s \wedge$ 
         $Y_s \leq \text{Normal } t\} \cap p_{\text{space}} p$ ) +
      prob p ( $\{s \mid X_d s < Y_s \wedge Y_s \leq \text{Normal } t\} \cap p_{\text{space}} p$ ))
```

[WSP_DFT_event]

```
 $\vdash \forall p \ Y X_a X_d t.$ 
  ( $\forall s.$   $0 \leq Y$  s)  $\wedge$  ( $\forall s.$  ALL_DISTINCT [ $X_a$  s;  $X_d$  s;  $Y$  s])  $\Rightarrow$ 
  (DFT_event p (WSP  $Y X_a X_d$ ) t =
    ( $\{s \mid$ 
       $Y_s < X_a s \wedge X_a s \leq \text{Normal } t \wedge 0 \leq Y_s \wedge$ 
       $Y_s \leq \text{Normal } t\} \cap p_{\text{space}} p$   $\cup$ 
      ( $\{s \mid X_d s < Y_s \wedge Y_s \leq \text{Normal } t\} \cap p_{\text{space}} p$ ))
```

