Formalizing Potential Flows using the HOL Light Theorem Prover

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Abstract. Potential flow is a theoretical model that describes the movement of a fluid, e.g., water or air in situations where viscosity and turbulence are assumed to be negligible. This type of flow is often used as an idealized model to describe the behavior of fluids in specific contexts, such as in fluid dynamics and aerodynamics. In this paper, we present a higher-order logic formalization of potential flows that are governed by the Laplace's equation. We focus on formally modeling fundamental flows such as the uniform, source/sink, doublet, and vortex flows in the HOL Light theorem prover. We then prove the validity of the exact solutions of the Laplace's equation for these types of flows. We also present the formal verification of the linearity of the Laplace's operator, which is essential for applying the superposition principle. In order to demonstrate the practical effectiveness of our formalization, we formally verify several applications such as rankine oval, flow past a circular cylinder and flow past a rotating circular cylinder, each of which involves combining these standard flows to model more complex fluid dynamics.

Keywords: Potential Flows, Partial Differential Equations, Laplace's Equation, Higher-Order Logic, Theorem Proving, HOL Light

1 Introduction

Potential flow theory is a key concept in the discipline of fluid dynamics. It uses harmonic functions to study a wide range of fluid-related phenomena within the theoretical framework of this field of study. Potential flow describes the velocity field as the gradient of a scalar function known as the velocity potential. Moreover, it characterizes the flow as irrotational and incompressible and provides valuable insights into fluid dynamics. This idealization is in close approximation to real-world scenarios of practical importance. For instance, in aerodynamics, this theory has played a pivotal role in developing analytical models to understand airflow around airfoils, wings, and related aerodynamic surfaces, which in turn facilitate the prediction of crucial aerodynamic forces such as lifts [12].

The foundation of addressing aerodynamic problems lies in the equations that govern the flow. While fluid motion is governed by the Navier-Stokes (NS) equations [17], which is a vector equation that includes three different scalar equations along with the conservation of the mass equation [18], their nonlinear nature renders them challenging to solve [13]. Consequently, the Laplace equation, which is a prevalent class of partial differential equations [16] emerges as a preferred alternative, providing an exact representation of incompressible, inviscid and irrotational flows. Unlike the NS equations, the use of the Laplace equation is much easier than using fully viscous NS equations. This equation forms the basis of potential flow theory, where both the stream function and velocity potential, as algebraic functions satisfying the Laplace's equation, can be combined to construct flow fields. Moreover, the superposition of basic potential flow solutions is a crucial step in the analysis of aerodynamic configurations. This method leverages the linearity of the Laplace equation, enabling for the construction of models that represent intricate scenarios by combining simpler flow elements [15].

Due to the fundamental importance of the Laplace equation in physics, applied mathematics, and engineering, numerous well-established analytical and numerical techniques exist for solving this equation, especially in the field of aerodynamics. These techniques are also useful in developing advanced computational methods for determining potential flows around the complex three-dimensional geometries common in modern aircraft design [12]. For instance, the method of images [8] are applied to model potential flows around airfoils and wings, where a combination of real and image sources helps satisfy the no-flow boundary conditions on solid surfaces. On the other hand, numerical techniques such as the panel methods [3] are computational models that simplify the assumptions concerning the aerodynamic principles and characteristics of airflow over an aircraft. Despite the prevalence of traditional techniques in analyzing aerodynamic problems, there exists a notable concern regarding their accuracy. For instance, paper-and-pencil methods carry a risk of human errors. It is possible that a mathematical result may be misapplied when using a manual method, as it is not possible to guarantee that all required assumptions are valid. In regard to simulation tools, the accuracy of simulation results depends on various factors, including the precision of numerical techniques, and computational issues may arise, especially in the context of large models.

In contrast, formal verification employs computer-based techniques for the mathematical modeling, analysis, and verification of abstract and physical systems. A prominent technique in formal verification is higher-order-logic (HOL) theorem proving [10], which is an interactive approach that involves human-machine collaboration for the development of correct proofs. Its expressive capabilities are sufficient for the description of the majority of classical mathematical theories, including differentiation, integration, higher transcendental functions, and topological spaces. Given the fundamental role of potential flow theory in the early stages of aircraft design, where it is used to predict the behavior of airflow around wings, the safety-critical nature of potential flow applications becomes evident. Therefore, it is imperative to employ robust verification tools that can ensure the accuracy and reliability of these theoretical models.

In this paper, we propose to use higher-order logic theorem proving for the formalization of standard potential flows that are governed by the Laplace's equation. We also provide the formal verification of these exact potential flow solutions along with their applications in aerodynamics.

The major contributions of the paper are:

- Higher-order logic formalization of real potential flows, namely, uniform, source/sink, doublet and vortex flows.
- Formal verification of the validity of these potential flow solutions for the Laplace's equation for aerodynamic applications
- Formal verification of the linearity of the Laplace's operator
- Formal verification of several applications built by superimposing these real flows

While there exist some formalization work of other types of partial differential equations, such as the Wave Equation [4], the Heat Equation [6] and the Telegrapher's Equation [7], to the best of our knowledge, there exists no formalization of the Laplace equation in the literature. Therefore, the formal analysis of potential flows governed by the Laplace's equation using HOL theorem proving is the first of its kind, which could be very useful for safety-critical applications.

The rest of the paper is organized as follows: Section 2 describes some preliminary details of the potential flow theory and the HOL Light theorem prover that are necessary for understanding the rest of the paper. We present the formalization of standard potential flows in Section 3. In Section 4, we provide the formal verification of the validity of the exact potential flow solutions for the Laplace's equation. Section 5 provides the formal verification of the linearity of the Laplace's operator as well as the verification of more complicated flows that are constructed by combining the standard potential flows. Finally, Section 6 concludes the paper.

2 Preliminaries

In this section, we briefly describe the HOL Light theorem prover as well as some of the associated functions and symbols that are necessary for understanding the rest of the paper. We also provide some background knowledge about potential flow theory.

2.1 HOL Light Theorem Prover

Interactive theorem proving is a collaborative process between a machine and a human user, where they work together interactively to generate a formal proof. The use of theorem proving systems is common in the verification of both software and hardware as well as in pure mathematics. For instance, a verification engineer can manually build a logical model of the system and subsequently verify the desired properties while providing guidance to the theorem proving tool. Similarly, a mathematician can use theorem provers in the verification of standard pure mathematical contexts. HOL Light [11], developed by Harrison, is one of the theorem provers in the HOL family [10], characterized by its small logical kernel. In HOL Light, the process of proving a theorem begins with the user entering the theorem's statement as the goal in a new proof. The proofs in HOL Light rely on tactics that break down complex goals into more straightforward subgoals. Furthermore, HOL Light provides a variety of automated proof procedures and proof assistants to assist users in guiding and completing their proofs. In addition, users have the flexibility to craft and implement their own personalized automation methods.

Table 1 provides the mathematical interpretations of some of the HOL Light symbols and functions used in this paper.

Table 1: HOL Light Symbols

HOL Light Symbols	Standard Symbols	Description
`&a`	$\mathbb{N} \to \mathbb{R}$	Type casting from Natural numbers to Reals
`&num`	$\{0, 1, 2..\}$	Positive Integers data type
`λx.t`	$\lambda x.\ t$	Function that maps x to $t(x)$
`real`	$\mathbb{R}$	Real data type
`@f`	Hilbert choice operator	Returns f if it exists
`atreal x`	Real net	At real variable x
`--x`	$-x$	Unary negation of x

2.2 Brief Review of Potential Flow Theory

Potential flow can be defined as steady, incompressible and irrotational flow. A condition that is necessary and sufficient to identify a flow as irrotational:

$$\overrightarrow{\nabla} \times \overrightarrow{V} = 0 \tag{1}$$

This indicates that the velocity field **V** is a conservative vector field denoted by the gradient of a scalar velocity potential function (ϕ):

$$\overrightarrow{V} = \overrightarrow{\nabla}\phi \tag{2}$$

If the velocity potential is known, then the velocity at any point can be determined using

$$u = \frac{\partial \phi}{\partial x}, \quad v = \frac{\partial \phi}{\partial y} \tag{3}$$

The irrotationality condition for two-dimensional flows vorticity is given by:

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \xi \tag{4}$$

Here, $\xi = 0$ since the flow is irrotational.

Similarly, in the case of an incompressible flow, it follows from the continuity equation that:

$$\vec{\nabla}.\vec{V} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0 \tag{5}$$

The two-dimensional continuous flow is described by the stream function (for incompressible flow) ψ, which determines the velocity at any point as:

$$u = \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial x} \tag{6}$$

Substituting Equations (3) and (6) into Equations (5) and (4), respectively, yields the conditions for continuous irrotational flow:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \tag{7}$$

which is Laplace's equation in Cartesian coordinates [12]. It can also be written in polar coordinates as:

$$\nabla^2 \psi = \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2} \tag{8}$$

where the operator nable squared

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

is referred to as the Laplacian operator.

Both the velocity potential (ϕ) and the stream function (ψ) are employed to describe the flow field in fluid dynamics and they satisfy the Laplace's equation. There are notable similarities and differences between the stream function and the velocity potential. For instance, while the stream function can be employed to describe both rotational and irrotational flows, the velocity potential is only defined for irrotational flow. On the other hand, the velocity potential is applicable to three-dimensional flows, whereas the stream function has only been defined for two-dimensional flows.

There are several techniques available to determine both the velocity potential (ϕ) and the stream function (ψ). For instance, commmon numerical and analytical techniques such as Finite Element Method (FEM) [5] and seperation of variables [9], respectively are frequently used to solve Laplace's equation with the appropriate boundary conditions. Another popular technique is to find some simple functions that satisfy the Laplace's equation and to model the flow around the body of interest, which is possible due to the linearity of the Laplace's equation. The focus of this paper will be this latter method, which is the most widely used procedure for potential flows. In the next section, we will present the formalization of these basic flows.

3 Formalizing Standard Potential Flow Solutions

In this section, we present some basic functions which satisfy the Laplace's equation. Any function that satisfies this equation describes a potential flow. It is noteworthy that in this work, we are interested in employing exact potential flow solutions to formally validate them for the Laplace's equation. Furthermore, our objective is to use these elementary flows as building blocks to construct a desired flow field, rather than deriving them.

3.1 Uniform Flow

The most basic type of flow is a uniform steady flow as shown in Figure 1. A uniform flow directed in the positive x-direction has the velocity components $u = U$ and $v = 0$ everywhere. This type of flow is irrotational and therefore possesses a velocity potential ϕ, which can be shown as follows:

$$\phi = Ux \tag{9}$$

Additionally, the stream function can be expressed as:

$$\psi = Uy \tag{10}$$

The formal representation of a uniform flow for the stream function is given as follows:

Definition 1. *Uniform Flow*

```
⊢def ∀U y. stream_uniform U y = U * y
```

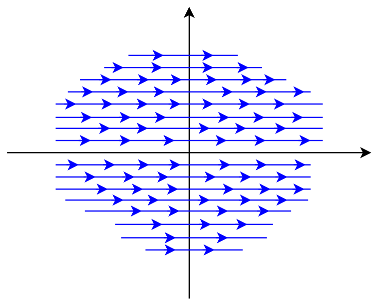
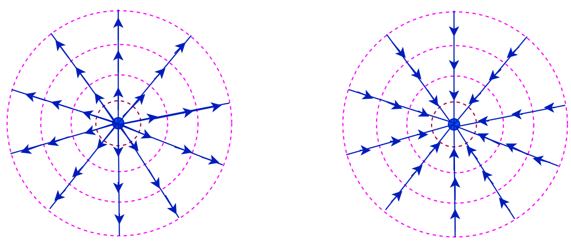


Fig. 1: Uniform Flow

3.2 Source/Sink Flow

In two-dimensional fluid dynamics, a source is defined as a point where fluid propagates radially outward, while a sink represents a point of negative source characterized by inward radial fluid movement as illustrated in Figure 2(a) and 2(b), respectively.



(a) Source flow (b) Sink flow

Fig. 2: Source/Sink Flow

The exact potential flow solutions centered at point (x_0,y_0) for the stream function and the velocity potential are mathematically expressed as [12]:

$$\psi(x,y) = \frac{m}{2\pi} tan^{-1}\left(\frac{y-y_0}{x-x_0}\right) \tag{11}$$

$$\phi(x,y) = \frac{m}{4\pi} In((x-x_0)^2 + (y-y_0)^2) \tag{12}$$

Here, m denotes the strength of the source. A positive m ($m > 0$) denotes a source flow, whereas a negative m ($m < 0$) indicates a sink flow.

Now, we formalize the above equations, i.e., Equations (11) and (12) in HOL Light as follows:

Definition 2. *Source Flow for the Stream Function*
```
⊢def ∀m x y x0 y0.
 stream_source m x y x0 y0 =
    m / (&2 * pi) * atn ((y - y0) / (x - x0))
```

Definition 3. *Source Flow for the Velocity Potential*
```
⊢def ∀m x y x0 y0.
 velocity_source m x y x0 y0 =
    m / (&4 * pi) * log ((x - x0) pow 2 + (y - y0) pow 2)
```

Here, `atn` and `log` indicate the inverse of the tangent function and the natural logarithm, respectively.

In the next subsections, we will use the polar coordinates r and θ to describe the doublet and vortex flows. Note that uniform and source/sink flows can be similarly represented using polar coordinates, utilizing the relationships $x = r\cos\theta$, $y = r\sin\theta$. These transformations are particularly useful for practical examples.

3.3 Doublet Flow

As depicted in Figure 3, the doublet is a special flow pattern that arises when a source and a sink of equal strength are constrained to have a constant ratio of strength to distance (κ), as the distance approaches zero.

The resulting solutions for the stream function and the velocity potential are as follows:

$$\psi(r,\theta) = -\frac{\kappa}{2\pi r} sin\theta \tag{13}$$

$$\phi(r,\theta) = \frac{\kappa}{2\pi r} cos\theta \tag{14}$$

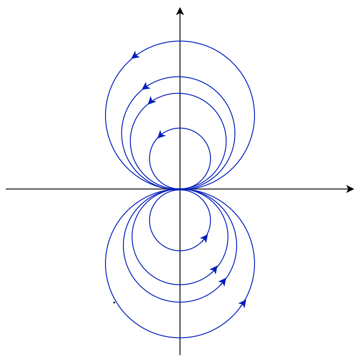


Fig. 3: Doublet Flow

The next step is to formalize the above equations (Equations (13) and (14)) in HOL Light:

Definition 4. *Doublet Flow for the Stream Function*

```
⊢def ∀K theta r.
  stream_doublet K theta r = --(K / (&2 * pi * r)) * sin (theta)
```

Definition 5. *Doublet Flow for the Velocity Potential*

```
⊢def ∀K theta r.
 velocity_doublet K theta r = (K / (&2 * pi * r)) * cos (theta)
```

where `stream_doublet` and `stream_doublet` accept the strength `K`, the radius `r` and the angle `theta` and return the corresponding functions.

3.4 Vortex Flow

A two-dimensional, steady flow that circulates about a point is known as a line vortex. In this type of flow, the streamlines form concentric circles around a specific point as shown in Figure 4.

It is important to note that the irrotational nature of the flow is not contradicted by the potential vortex formulation. Fluid elements travel in a circular path around the vortex centre without rotating about their axes, thus meeting the condition of irrotational flow.

The exact potential flow solution centered at the origin is mathematically expressed as:

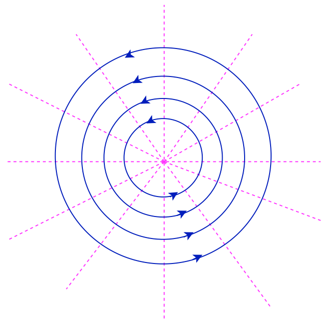


Fig. 4: Vortex Flow

$$\psi(r,\theta) = \frac{\Gamma}{2\pi} In(r) \tag{15}$$

$$\phi(r,\theta) = -\frac{\Gamma}{2\pi}\theta \tag{16}$$

where Γ represents the circulation, which is often positive when moving counterclockwise.

Next, we formalize the vortex flow for the stream function and the velocity potential, i.e., Equations (15) and (16) as:

Definition 6. *Vortex Flow for the Stream Function*

```
⊢def ∀gamma r.
 stream_vortex gamma r = gamma / (&2 * pi) * log (r)
```

Definition 7. *Vortex Flow for the Velocity Potential*

```
⊢def ∀gamma theta.
 velocity_vortex gamma theta = --gamma / (&2 * pi) * theta
```

Table 2 summarizes the potential flows that are presented in this section.

Table 2: Standard Flows Overview

Flow Type	Stream Function	Velocity Potential
Uniform flow in the x-direction	$\psi(x,y) = Uy$	$\psi(x,y) = Ux$
Source/Sink	$\psi(x,y) = \frac{m}{2\pi} tan^{-1}\left(\frac{y-y_0}{x-x_0}\right)$	$\phi(x,y) = \frac{m}{4\pi} In((x-x_0)^2 + (y-y_0)^2)$
Doublet	$\psi(r,\theta) = -\frac{\kappa}{2\pi r} sin\theta$	$\phi(r,\theta) = \frac{\kappa}{2\pi r} cos\theta$
Vortex	$\psi(r,\theta) = \frac{\Gamma}{2\pi} In(r)$	$\phi(r,\theta) = -\frac{\Gamma}{2\pi}\theta$

4 Formal Verification of the Laplace Equation's Solutions

In this section, we present the formal verification of the exact potential flow solutions of the Laplace's equation. The purpose of this verification is to ensure the correctness of analytical solutions and then establish their foundational role in describing fluid behavior and facilitating engineering applications.

For this verification, our first step is to formalize the Laplace's equation in both Cartesian and polar coordinates in the HOL Light as follows [1]:

Definition 8. *The Laplace's Equation in Cartesian Coordinates*

```
⊢def laplace_equation psi(x,y) ⇔ laplace_operator psi(x,y) = &0
```

where Laplace equation accepts the real function `psi`: $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$, the space variables `x`:$\mathbb{R}$ and `y`:$\mathbb{R}$ and returns the corresponding Laplace's equation. The function Laplace operator is formalized as:

Definition 9. *Laplace Operator*

```
⊢def ∀psi x y.
laplace_operator psi(x,y) =
   higher_real_derivative 2 (λx. psi(x,y)) x +
     higher_real_derivative 2 (λy. psi(x,y)) y
```

[1] Here, we present the formalizations for the stream function for brevity. The verification presented in this section was also done for the velocity potential.

Here, higher_real_derivative represents the n^{th}-order real derivative of a function.

The formal representation of the Laplace's equation in polar coordinates, i.e., Equation (8) is given as follows:

Definition 10. *The Laplace's Equation in Polar Coordinates*

```
⊢def ∀psi r theta.
laplace_in_polar psi r theta =
  higher_real_derivative 2 (λr. psi(r,theta)) r +
   &1/r * higher_real_derivative (λr. psi(r,theta)) r +
    &1/(r pow 2) * higher_real_derivative (λtheta. psi(r,theta)) theta = &0
```

where the HOL Light function laplace_in_polar mainly accepts the function psi of type $\mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$, the radial distance r and the angle theta and returns the corresponding equation. We can also formalize the Laplace's equation for the velocity potential in a similar manner.

With the formal definitions outlined previously, an important step is to verify that these potential flow solutions satisfy the Laplace's equation. In other words, this is the main condition for potential flows to be valid, which is fundamental for understanding fluid behavior in various contexts. We start with the verification of the source flow for the stream function, i.e., Equation (11) in HOL Light as follows:

Theorem 1. *Verification of Source Flow for the Stream Function*

```
⊢thm ∀m x0 y0 psi.
[A1] (∀x. x ≠ x0) ∧ [A2] (∀y. y ≠ y0) ∧
[A3] (∀x y. psi(x,y) = stream_source m x y x0 y0)

  ⇒ stream_laplace psi x y
```

Assumptions A1 and A2 ensure that the points in a Cartesian coordinate system are different than each other. Assumption A3 provides the solution of the Laplace's equation for source flow, i.e., Equation (11). The proof of the above theorem is mainly based on the real differentiation of the source flow solution with respect to the parameters x and y.

Our next step is to formally verify the doublet flow (Equation (13)) as the following HOL Light theorem:

Theorem 2. *Verification of Doublet Flow for the Stream Function*

```
⊢thm ∀K u.
[A1] (λr. &0 < r) ∧
[A2] (∀r theta. psi(r,theta) = stream_doublet K theta r))

  ⇒ laplace_in_polar psi r theta
```

Assumption A1 is required to ensure that the radial distance is greater than zero. Assumption A2 provides the solution of the Laplace's equation in polar coordinates (Equation (8)) for doublet flow (Equation (13)).

The verification of Theorem 2 is mainly based on the properties of real derivative [1] and some real arithmetic reasoning.

Finally, the vortex flow, i.e., Equation (15) is verified as the following theorem:

Theorem 3. *Verification of Vortex Flow for the Stream Function*

```
⊢thm ∀gamma u.
[A1] (λr. &0 < r) ∧
[A2] (∀r theta. psi(r,theta) = stream_vortex gamma u r theta))

  ⇒ laplace_in_polar psi r theta
```

Assumption A1 is the same as that of Theorem 2. The conclusion of Theorem 3 provides that the vortex flow solution satisfies the Laplace's equation. The proof of Theorem 3 is primarily based on the real differentiation of the vortex flow solution with respect to the parameters r and theta.

In the next section, we use these formally verified solutions to build more complicated flows which are widely applied in the analysis of flow patterns around an airfoil [14].

5 Applications of Standard Flows

The Laplace's equation is a second-order, linear, eliptic partial differential equation. Thanks to the linearity of the Laplace's equation, more complicated flow fields can be constructed from the superposition of basic solutions. If ψ_1 and ψ_2 are the solutions (stream functions) of the Laplace's equation and then their linear combination $\psi_1 + \psi_2$ will also be a solution for a two-dimensional incompressible and irrotational flow. This unique feature makes this equation a powerful tool to analyze fluid flow problems. The ability to obtain new flow patterns by superimposing known flows is fundamental to wing theory, as it provides simple solutions to complex problems [2].

Our first step is to formally verify the linearity of the Laplace operator due to its importance for the superposition principle.

Theorem 4. *Linearity of Laplace Operator*

```
⊢thm ∀psi phi a b.
[A1] (∀x. (λx. psi(x,y)) real_differentiable atreal x) ∧
[A2] (∀x. (λx. phi(x,y)) real_differentiable atreal x) ∧
[A3] (∀y. (λy. psi(x,y)) real_differentiable atreal y) ∧
[A4] (∀y. (λy. phi(x,y)) real_differentiable atreal y) ∧
[A5] (∀x. (λx. real_derivative (λx. psi(x,y)) x)
          real_differentiable atreal x) ∧
[A6] (∀x. (λx. real_derivative (λx. phi(x,y)) x)
          real_differentiable atreal x)
[A7] (∀y. (λy. real_derivative (λx. psi(x,y)) y)
          real_differentiable atreal y)
[A8] (∀y. (λy. real_derivative (λy. phi(x,y)) y)
          real_differentiable atreal y)
```

```
⇒ laplace_operator (λ(x,y). a * psi(x,y) + b * phi(x,y)) (x,y) =
      a * laplace_operator (λ(x,y). psi(x,y)) (x,y) +
        b * laplace_operator (λ(x,y). phi(x,y)) (x,y)
```

Assumptions `A1` and `A2` ensure that the real-valued functions `psi` and `phi` are differentiable at `x`, respectively. Assumptions `A3` and `A4` assert the differentiability of the functions `psi` and `phi` at `y`, respectively. Additionally, Assumptions `A5` and `A6` provide the differentiability conditions for the derivatives of the functions `psi` and `phi` at `x`, respectively. Similarly, Assumptions `A7` and `A8` guarantee the differentiability conditions for the derivatives of the functions `psi` and `phi` at `x`, respectively. The proof of the theorem above relies mainly on the properties of derivatives and the differentiability of real-valued functions.

5.1 Rankine Oval

By combining the exact solutions for uniform and source/sink flows, we can construct a flow field around an oval-shaped object. The resultant configuration is known as a Rankine oval.

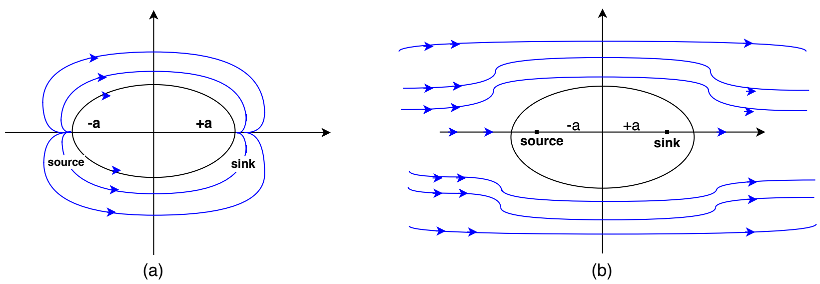


Fig. 5: Rankine Oval [13]

We start by analyzing the flow pattern around a source and a sink. The source and sink are placed along the x-axis, separated by a distance of $2a$, as depicted in Figure 5(a). The origin is situated equidistantly between them. We now superimpose the uniform, source and sink flows, all positioned in the x-direction, with a line source located at $(-a, 0)$ and a line sink of equal and opposite strength located at $(+a, 0)$, as depicted in Figure 5(b). Assume the strengths of these source and the sink are $+m$ and $-m$, respectively. The overall stream function (ψ) and velocity potential (ϕ) for this combination of flows are expressed as:

$$\psi = \psi_{uniform} + \psi_{source} + \psi_{sink} \tag{17}$$

$$\phi = \phi_{uniform} + \phi_{source} + \phi_{sink} \tag{18}$$

Mathematically, they are represented by the combination of Equations (9), (10), (11) and (12) for the stream function and the velocity potential:

$$\psi(x,y) = -Uy + \frac{m}{2\pi}\left[arctan\left(\frac{y}{x+a}\right) - arctan\left(\frac{y}{x-a}\right)\right] \tag{19}$$

$$\phi(x,y) = Ux + \frac{m}{4\pi} In\left(\frac{(x+a)^2+y^2}{(x-a)^2+y^2}\right) \tag{20}$$

Next, we formally verify these combined flows for the stream function as the following HOL Light theorem[2]:

Theorem 5. *Verification of Rankine Oval for the Stream Function*

```
⊢thm ∀U m a psi x0 x1 y0 y1.
[A1] (∀x. x ≠ a) ∧ [A2] (∀x. x ≠ --a) ∧ [A3] x0 = --a ∧
[A4] x1 = a ∧ [A5] y0 = &0 ∧ [A6] y1 = &0 ∧
[A7] (∀x y. psi(x,y) = sum (0..2) (λn. EL n [--stream_uniform U y;
                   stream_source m x y x0 y0; stream_sink m x y x1 y1]))
  ⇒ stream_laplace psi x y
```

Assumptions A1 and A2 guarantee that the validity of our expression by specifying that x must be different from a and --a, respectively. Asssumptions A3 and A4 provide the distance from the origin. Assumption A5 and A6 assert that the points y0 and y1 are equal to zero since the flows are oriented in towards the x-direction. Assumption A7 provides the combined solutions for the stream function, i.e., Equation (19). Here, the function EL n l extracts the n^{th} element from a list l. The verification of Theorem 5 is mainly based on the properties of real derivatives, some real arithmetic reasoning and the following HOL Light lemma:

Lemma 1. *Superposition of the Solutions*

```
⊢lem ∀U m x y x0 x1 y0 y1.
sum (0..2) (λn. EL n [--stream_uniform U y; stream_source m x y x0 y0;
 stream_sink m x y x1 y1]) = --stream_uniform U y + stream_source m x y x0 y0
                            + stream_sink m x y x1 y1
```

The above lemma states that the summation of the list equals to the linear combination of uniform, source and sink flows.

[2] Here, we only present the verification of applications that are provided in this section for the stream function, for brevity. Additionally, we conducted a formal verification for the velocity potential as well.

5.2 Potential Flow Past a Circular Cylinder

As shown in Figure 6, we can build a potential flow solution for the flow around a circular cylinder using the superposition of a uniform and a doublet flow in the x-direction. The resulting stream function and velocity potential for this particular combination of potential flows can be given as:

$$\psi = \psi_{uniform} + \psi_{doublet} \tag{21}$$

$$\phi = \phi_{uniform} + \psi_{doublet} \tag{22}$$

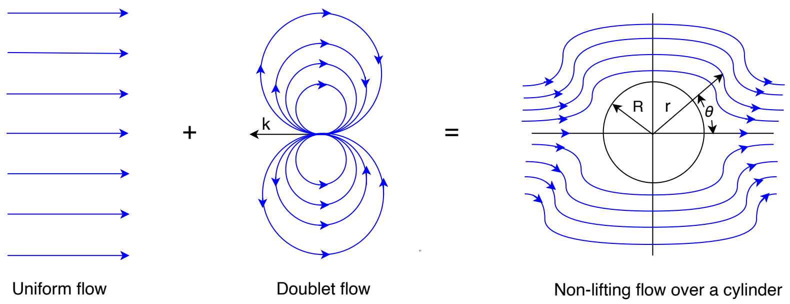


Fig. 6: Potential Flow Past a Circular Cylinder [13]

We can mathematically express this combination by adding the solutions for uniform and doublet flow, i.e., Equations (9), (10), (13) and (14). It is known that $y = rsin\theta$ in polar coordinates.

$$\psi(r,\theta) = U\left(r + \frac{\kappa}{2\pi r}\right) sin\theta \tag{23}$$

$$\phi(r,\theta) = U\left(r - \frac{\kappa}{2\pi r}\right) cos\theta \tag{24}$$

Next, we formally verify Equation (23) in HOL Light as follows:

Theorem 6. *Verification of Potential Flow Past a Circular Cylinder*

```
⊢thm ∀U K y psi.
[A1] (∀r. &0 < r) ∧ [A2] (∀r theta. y = r * sin(theta)) ∧
[A3] (∀r theta. psi(r,theta) = sum (0..1) (∀n. EL n [stream_uniform U y;
                                          stream_doublet K theta r]))
  ⇒ laplace_in_polar psi r theta
```

Assumption `A1` ensures that the radial distance is greater than zero, while Assumption `A2` indicates that `y = r * sin(theta)` in polar coordinates. Assumption `A3` provides the superposition of the uniform and doublet flow solutions for the stream function, i.e., Equation (23). Similar to Theorem 5, we proved a lemma regarding superposition of the solutions as well as proving the real derivatives of the solution in order to formally verify this theorem.

5.3 Potential Flow Past a Rotating Circular Cylinder

The flow around a rotating circular cylinder can be constructed by combining a doublet flow, a vortex flow, and a uniform flow using superposition as shown in Figure 7. In other words, the stream function and the velocity potential for this combination of potential flows can be given as:

$$\psi = \psi_{uniform} + \psi_{doublet} + \psi_{vortex} \tag{25}$$

$$\phi = \phi_{uniform} + \phi_{doublet} + \phi_{vortex} \tag{26}$$

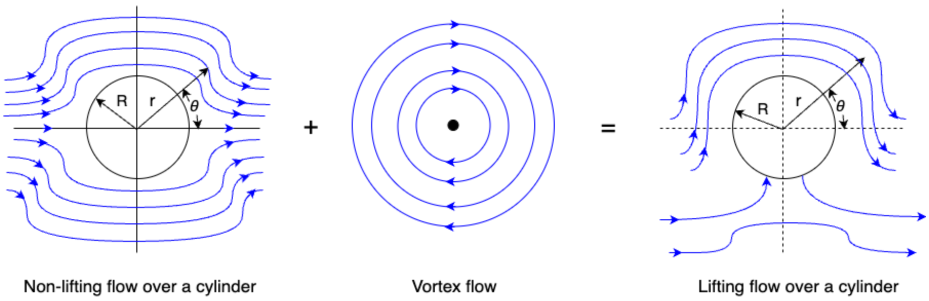


Fig. 7: Potential Flow Past a Rotating Circular Cylinder [13]

It is important to note that combining a uniform flow and a doublet flow effectively models the flow around a non-rotating circular cylinder, as given by Equations (23) and (24). Therefore, we can write the final mathematical expression of these flows for the stream function and the velocity potential by adding the solutions, i.e., Equations (15), (16), (23) and (24) as:

$$\psi(r, \theta) = U\left(r + \frac{\kappa}{2\pi r}\right) sin\theta + \frac{\Gamma}{2\pi} In(r) \tag{27}$$

$$\phi(r, \theta) = U\left(r - \frac{\kappa}{2\pi r}\right) cos\theta + -\frac{\Gamma}{2\pi}\theta \tag{28}$$

The above equations can be alternatively written as:

$$\psi(r, \theta) = Ursin\theta\left(1 - \frac{R^2}{r^2}\right) + \frac{\Gamma}{2\pi} Inr \tag{29}$$

$$\phi(r, \theta) = Urcos\theta\left(1 - \frac{R^2}{r^2}\right) + \frac{\Gamma}{2\pi}\theta \tag{30}$$

where $R^2 = \dfrac{m}{2\pi U}$ and m is the strength of the doublet.

Finally, we formally verify Equation (27) as the following HOL Light theorem:

Theorem 7. *Verification of Potential Flow Past a Rotating Circular Cylinder*

```
⊢thm ∀U K y gamma psi.
[A1] (∀r. &0 < r) ∧ [A2] (∀r theta. y = r * sin(theta)) ∧
[A3] (∀r theta. psi(r,theta) = sum (0..2) (∀n. EL n [stream_uniform U y;
                  stream_doublet K theta r; stream_vortex gamma theta r]))
  ⇒ laplace_in_polar psi r theta
```

The assumptions A1-A2 are the same as those of Theorem 6. Assumption A3 provides the combination of the uniform, doublet and vortex flow solutions for the stream function, i.e., Equation (27). The verification of Theorem 7 is similar to that of Theorem 6.

5.4 Discussion

Potential flow theory is a unique field at the intersection of mathematical physics and aerodynamics, driven primarily by its practical applications. This theory is extensively employed in aerodynamics to model and analyze potential flows, which describe the behavior of inviscid and incompressible fluids. A notable aspect of the work presented in this paper is the development of the first formalization of potential flows which has wide applications in aerodynamics, particularly in airfoil theory. One of the main challenges of this work is its interdisciplinary nature, as it requires a deep understanding of aerodynamic principles, the integration of mathematics, and the meticulous process of interactive theorem proving. Another significant challenge is verifying exact analytical solutions governed by the Laplace's equation. The proof process must establish the real derivatives of these solutions and their linear combinations. While traditional paper-and-pencil proofs can overlook trivial details, theorem proving demands a substantial amount of time due to the undecidable nature of higher-order logic and requires every detail to be meticulously provided to the computer. One of the benefits of this work is that it addresses these challenges by formalizing the core concepts of potential flow theory, allowing available results to be built upon to minimize user interaction. Additionally, all of the verified theorems and lemmas are general, opening the door to future expansions. We also believe that our work can be a significant step towards bridging the gap between theorem proving and the aerospace engineering communities, thereby enhancing its applicability in industrial settings.

6 Conclusion

In this paper, we conducted the formal specification and verification of standard potential flows solutions which satisfy the Laplace's equation using higher-order logic theorem proving. We first formalized four fundamental potential flows, namely, the uniform, source/sink, doublet and vortex flows. Moreover, we formally modeled the Laplace's equation in both Cartesian and polar coordinates.

Furthermore, we formally verified the linearity of the Laplace's equation since it is a very powerful tool to create more complicated flow fields. We then constructed the formal proof for the exact potential flow solutions of the Laplace's equation. Finally, in order to demonstrate the applicability of our formalization work, we formally analyzed several practical applications, including rankine oval, potential flow past a circular cylinder and a potential flow past a rotating circular cylinder. For the future work, we plan to extend our formalization for other complex-valued potential flows in order to analyze more complicated problems in aerodynamics.

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